

EXPONENTIAL CONVERGENCE OF SOLUTIONS FOR RANDOM HAMILTON-JACOBI EQUATIONS

RENATO ITURRIAGA, KONSTANTIN KHANIN, AND KE ZHANG

ABSTRACT. We show that for a family of randomly kicked Hamilton-Jacobi equations on the torus, almost surely, the solution of an initial value problem converges exponentially fast to the unique stationary solution. Combined with the results in [6] and [8], this completes the program started in [4] for the multi-dimensional setting.

1. INTRODUCTION

We consider the randomly forced Hamilton-Jacobi equation on the d dimensional torus

$$\partial_t \psi(x, t) + \frac{1}{2} (\nabla \psi(x, t) + b)^2 + F^\omega(x, t) = 0, \quad x \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d, \quad (1.1)$$

where $b \in \mathbb{R}^d$, ∇ stands for gradient in x , and F^ω is a random potential. By writing $u(x, t) = \nabla \psi(x, t)$, we obtain the stochastic Burgers equation

$$\partial_t u + (u \cdot \nabla) u = f^\omega(y, t), \quad y \in \mathbb{R}^d, t \in \mathbb{R}, \quad (1.2)$$

$f^\omega(y, t) = -\nabla F^\omega(y, t)$ with the condition $\int u(x, t) dx = b$. This is one of the motivations of our study. On the other hand, (1.1) is a particular example of the more general Hamilton-Jacobi equation

$$\partial_t \psi + H^\omega(x, \nabla \psi) = 0 \quad (1.3)$$

where $H^\omega(x, p)$ is strictly convex and superlinear in p (called the Tonelli Hamiltonians). Many of our results can be generalized to (1.3), but we will restrict to (1.1) for simplicity.

We are interested in two types of random potentials. In [4], the authors consider the dimension $d = 1$, with the “white noise potential”

$$F^\omega(y, t) = \sum_{i=1}^M F_i(y, t) = \sum_{i=1}^M F_i(y) \dot{W}_i(t), \quad (1.4)$$

where $F_i : \mathbb{T}^d \rightarrow \mathbb{R}$ are smooth functions, and \dot{W}_i are independent white noises. It is shown that finite time solutions of (1.1) converges exponentially fast to a unique stationary solution. In this paper, we generalize this result to arbitrary dimensions, for a related “kicked” model.

The “kicked force” model was introduced in [6], with

$$F^\omega(y, t) = \sum_{j \in \mathbb{Z}} F_j^\omega(y) \delta(t - j), \quad (1.5)$$

where F_j^ω is an i.i.d. sequence of potentials, and $\delta(\cdot)$ is the delta function. We focus on the “kicked” potential (1.5) as it is simpler, but retains most of the features of the system.

The system (1.3) does not admit classical solutions in general, and the solution is interpreted using the Lax-Oleinik variational principle. There is a semi-group family of operators (see (2.2))

$$K_{s,t}^{\omega,b} : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d),$$

such that the function $\psi(x, \tau) = K_{s,\tau}^{\omega,b} \varphi(x)$, $s \leq \tau \leq t$ is the solution to (1.1) on the interval $[s, t]$ with the initial condition $\psi(x, s) = \varphi(x)$.

It is shown in [6] that under suitable conditions on the kicked force, almost surely, the system (1.1) admits a unique solution $\psi_\omega^-(x, t)$ (up to an additive constant) on the interval $(-\infty, \infty)$. Let us denote

$$\|\psi\|_* = \min_{C \in \mathbb{R}} \sup_{x \in \mathbb{T}^d} \|\psi(x) - C\|,$$

which is the suitable semi-norm for measuring convergence up to an additive constant. Then any solution on $[s, t]$ converges to ψ_ω^- as $s \rightarrow -\infty$, uniformly over all initial conditions in the semi-norm $\|\cdot\|_*$:

$$\lim_{s \rightarrow -\infty} \sup_{\varphi \in C(\mathbb{T}^d)} \|K_{s,t}^{\omega,b} \varphi(x) - \psi_\omega^-(x, t)\|_* = 0.$$

Our main result is that the above convergence is exponentially fast.

Main result. *There exists a (non-random) $\lambda > 0$ such that, almost surely,*

$$\limsup_{s \rightarrow -\infty} \frac{1}{|s|} \log \left(\sup_{\varphi \in C(\mathbb{T}^d)} \|K_{s,t}^{\omega,b} \varphi(x) - \psi_\omega^-(x, t)\|_* \right) < -\lambda,$$

see Theorem 1.

Remark. Exponential convergence is also known to hold in the viscous equation

$$\partial_t \psi + \frac{1}{2} (\nabla \psi)^2 + F^\omega = \nu \Delta \psi$$

(see [11]). However, in this case the a priori convergence rate $\lambda(\nu) \rightarrow 0$ as $\nu \rightarrow 0$. Since our result provides a non-zero lower bound on convergence rate when $\nu = 0$, it is an interesting question whether a uniform rate of convergence exists for the viscous equation.

The *a priori* convergence rate of the Lax-Oleinik semi-group is only polynomial in time, as evidence in the case when there is no force, i.e. $F^\omega = 0$. In the case when the force is non-random, exponential convergence is true when the Aubry set consists of finitely many hyperbolic periodic orbits or fixed points ([7]). According to a famous conjecture of Mañé, this condition holds for a generic force ([9]), however this conjecture is only proven when $d = 2$ among C^2 forces ([2], [3]).

In some sense, [8] proves a random version of Mañé’s conjecture. In the random case, the role of the Aubry set is taken by the globally minimizing orbit, and it is shown that this orbit is non-uniformly hyperbolic under the random Euler-Lagrange

flow. Conceptually, this hyperbolicity then allows the exponential convergence. However, this is quite delicate. To illustrate, let us outline the proof in the uniform hyperbolic case:

- (*Step 1*) Consider a solution $K_{-T,0}^{\omega,b}\varphi$ that is sufficiently close to the stationary solution $\psi^-(\cdot, 0)$, this is the case since we know the solution $K_{-T,0}^{\omega,b}\varphi \rightarrow \psi^-(\cdot, 0)$, albeit without any rate estimates.
- (*Step 2*) Show that the associated finite time minimizers is close to the Aubry set when $t \in [-2T/3, -T/3]$. By hyperbolic theory, any orbit that stays in a neighborhood of an hyperbolic orbit for time $T/3$ must be exponentially close to it at some point.
- (*Step 3*) Finite time minimizer being exponentially close to the Aubry set implies the solution is in fact exponentially close to ψ^- .

In the non-uniform hyperbolic case, Step 2 fails, because a non-uniform hyperbolic orbit only influence nearby orbits in a *random* neighborhood whose size changes from iterate to iterate. We are forced to devise a much more involved procedure:

- (1) (*Step A*) Reduce the problem to a local one, where we only study the solution in a small (random) neighborhood of the global minimizer.
- (2) (*Step B*) Consider a solution $K_{-T,0}^{\omega,b}\varphi$ is δ -close to the stationary solution $\psi^-(\cdot, 0)$ locally. Use a combination of variational and non-uniform hyperbolic theory to show that the finite time minimizer is δ^q -close to the global minimizer at some time, where $q > 1$. This step can only be done up to an exponentially small error.
- (3) (*Step C*) Use step B to show the solution $K_{-T,0}^{\omega,b}\varphi$ is δ^q -close to the stationary solution. Feed the new estimates into Step B, and repeatedly upgrade until δ is exponentially small.

We now present the outline of the paper. We formulate our assumptions and main result in Section 2. Basic properties of the viscosity solutions and stationary solutions are introduced in Sections 3 and 4. In Section 5, we reduce the main result to its local version, as outlined in Step A. This is Proposition 5.1.

In Section 6, we describe the upgrade procedure outlined in Step C. Step B is formulated in Proposition 6.1, and the proof is postponed to Sections 7 and 8.

2. STATEMENT OF THE MAIN RESULT

Consider the kicked potentials (1.5), where the random potentials F_j^ω are chosen independently from a distribution $P \in \mathcal{P}(C^{2+\alpha}(\mathbb{T}^d))$, with $0 < \alpha \leq 1$.

Given an absolutely continuous curve $\zeta : [s, t] \rightarrow \mathbb{T}^d$, we define the action of ζ to be

$$\mathbb{A}^{\omega,b}(\zeta) = \int_s^t \frac{1}{2} \left(\dot{\zeta}^2(\tau) - b \cdot \dot{\zeta} \right) d\tau - \sum_{s \leq j < t} F_j^\omega(\zeta(j)).$$

In other words, when s, t are integers, we include the kick at time s , but not at time t . For $0 < s < t \in \mathbb{R}$, and $x, x' \in \mathbb{T}^d$, the action function is

$$A_{s,t}^{\omega,b}(x, x') = \inf_{\zeta(s)=x, \zeta(t)=x'} \mathbb{A}^{\omega,b}(\zeta), \quad (2.1)$$

where ζ is absolutely continuous. The action function is Lipschitz in both variables.

The backward Lax-Oleinik operator $K_{s,t}^{\omega,b} : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ is defined as

$$K_{s,t}^{\omega,b} \varphi(x) = \min_{y \in \mathbb{T}^d} \left\{ \varphi(y) + A_{s,t}^{\omega,b}(y, x) \right\}. \quad (2.2)$$

We take (2.2) as the definition of our solution on $[s, t]$ with initial condition $\varphi(x)$. Due to the fact that $F^\omega(x, t)$ vanishes at non-integer times, $K_{s,t}^{\omega,b}$ is completely determined by its value at integer times. In the sequel we consider only $s = m, t = n \in \mathbb{Z}$. The operators satisfies a semi-group property: for $s < t < u$,

$$K_{t,u}^{\omega,b} K_{s,t}^{\omega,b} \varphi(x) = K_{s,u}^{\omega,b} \varphi(x)$$

We now state the conditions on the random potentials. The following assumptions are introduced in [6], which guarantees the uniqueness of the stationary solution.

Assumption 1. For any $y \in \mathbb{T}^d$, there exists $G_y \in \text{supp } P$ s.t. G_y has a maximum at y and that there exists $\delta > 0$ such that

$$G_y(y) - G(x) \geq \delta |y - x|^2.$$

Assumption 2. $0 \in \text{supp } P$.

Assumption 3. There exists $G \in \text{supp } P$ such that G has a unique maximum.

The following is proved in [6] under the weaker assumption that $F_j^\omega \in C^1(\mathbb{T}^d)$:

Proposition 2.1. [6]

(1) Assume that assumption 1 or 2 holds. For a.e. $\omega \in \Omega$, we have the following statements.

(a) There exists a Lipschitz function $\psi^-(x, n)$, $n \in \mathbb{Z}$, such that for any $m < n$,

$$K_{m,n}^{\omega,b} \psi^-(x, m) = \psi^-(x, n).$$

(b) For any $n \in \mathbb{Z}$, we have

$$\lim_{m \rightarrow -\infty} \sup_{\varphi \in C(\mathbb{T})} \|K_{m,n}^{\omega,b} \varphi(x) - \psi^-(x, n)\|_* = 0.$$

(2) Assume that assumption 3 holds. Then the conclusions for the first case hold for $b = 0$.

We now restrict to a specific family of kicked potentials. The following assumption is introduced in [6].

Assumption 4. Assume that

$$F_j^\omega(x) = \sum_{i=1}^M \xi_j^i(\omega) F_i(x), \quad (2.3)$$

where $F_i : \mathbb{T}^d \rightarrow \mathbb{R}$ are smooth non-random functions, and the vectors $\xi_j(\omega) = (\xi_j^i(\omega))_{i=1}^M$ is an i.i.d sequence of vectors in \mathbb{R}^M with an absolutely continuous distribution.

In [8], a stronger assumption is used to obtain information on the stationary solutions and the global minimizer. These additional structures provides the mechanism for exponential convergence. Let $\rho : \mathbb{R}^m \rightarrow \mathbb{R}$ be the density of ξ_j .

Assumption 5. Suppose assumption 4 holds, and in addition:

–

$$\mathbb{E}(|\xi_j|) = \int_{\mathbb{R}^M} |c| \rho(c) dc < \infty.$$

– For every $1 \leq i \leq M$, there exists non-negative functions $\rho_i \in L^\infty(\mathbb{R})$ and $\hat{\rho}_i \in L^1(\mathbb{R}^{M-1})$ such that

$$\rho(c) \leq \rho_i(c_i) \hat{\rho}_i(\hat{c}),$$

where $c = (c_1, \dots, c_M)$, $\hat{c}_i = (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_M)$.

Assumption 5 is rather mild. We only need to avoid the case that ρ is degenerate in some directions. In particular, it is satisfied if ξ_j^1, \dots, ξ_j^M are i.i.d. random variables with bounded densities and finite mean.

We now state the main theorem of this paper.

Theorem 1. (1) *Assume that assumption 5 and one of assumption 1 or 2 hold. Assume in addition that the mapping*

$$(F_1, \dots, F_M) : \mathbb{T}^d \rightarrow \mathbb{R}^M \tag{2.4}$$

is an embedding. For $b \in \mathbb{R}^d$, let ψ_ω^- be the unique stationary solution in Proposition 2.1. Then there exists a (non-random) $\lambda > 0$ and a random variables $N(\omega) > 0$ such that almost surely, for all $N > N(\omega)$,

$$\sup_{\varphi \in C(\mathbb{T}^d)} \|K_{m,n}^{\omega,b} \varphi - \psi_\omega^-(\cdot, n)\|_* \leq e^{-\lambda N}.$$

(2) *Assume that assumption 3 and 5 hold. Then the same conclusions hold for $b = 0$.*

3. VISCOSITY SOLUTIONS AND THE GLOBAL MINIMIZER

Let $I \subset \mathbb{R}$ be an interval. An absolutely continuous curve $\gamma : I \rightarrow \mathbb{T}^d$ is called a minimizer if for each interval $[s, t] \subset I$, we have $A_{s,t}^{\omega,b}(\gamma(s), \gamma(t)) = \mathbb{A}^{\omega,b}(\gamma|_{[s,t]})$. In particular, γ is called a forward minimizer if $I = (-\infty, t_0]$, a backward minimizer if $I = [s_0, \infty)$ and a global minimizer if $I = (-\infty, \infty)$.

Due to the kicked nature of the potential, a minimizer is always linear between integer times. Then any minimizer $\gamma : [m, n] \rightarrow \infty$ is completely determined by the sequence

$$x_j = \gamma(j), \quad v_j = \dot{\gamma}(j-), \quad m+1 \leq j \leq n. \tag{3.1}$$

The underlying dynamics for the minimizers is given by family of maps $\Phi_j^\omega : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{T}^d \times \mathbb{R}^d$

$$\Phi_j^\omega : \begin{bmatrix} x \\ v \end{bmatrix} \mapsto \begin{bmatrix} x + v - \nabla F_j^\omega(x) \bmod \mathbb{Z}^d \\ v - \nabla F_j^\omega(x) \end{bmatrix}, \quad (3.2)$$

The maps belong to the so-called *standard family*, and are examples of symplectic, exact and monotonically twist diffeomorphisms. For $m, n \in \mathbb{Z}$, $m < n$, denote

$$\Phi_{m,n}^\omega(x, v) = \Phi_{n-1}^\omega \circ \cdots \circ \Phi_m^\omega(x, v).$$

The (full) orbit of a vector (x_n, v_n) is given by the sequence

$$(x_j, v_j) = \Phi_{n,j}^\omega(x_n, v_n), \quad j > n, \quad (x_j, v_j) = (\Phi_{j,n}^\omega)^{-1}(x_n, v_n), \quad j < n.$$

If $\gamma : [m, n] \rightarrow \mathbb{T}^d$ is a minimizer, then (x_j, v_j) defined in (3.1) is an orbit, namely

$$\Phi_{j,k}^\omega(x_j, v_j) = (x_k, v_k), \quad m+1 \leq j < k \leq n.$$

In this case, we extend the sequence to $(x_m, v_m) = (\Phi_m)^\omega(x_{m+1}, v_{m+1})$ and call $(x_j, v_j)_{j=m}^n$ a minimizer.

The viscosity solution and the minimizers are linked by the following lemma:

Lemma 3.1 ([8], Lemma 3.2). (1) For $\varphi \in C(\mathbb{T}^d)$ and $m < n \in \mathbb{Z}$, for each $x \in \mathbb{T}^d$ there exists a minimizer $(x_j, v_j)_{j=m}^n$ such that $x_n = x$, and

$$K_{m,n}^{\omega,b} \varphi(x_n) = \varphi(x_m) + A_{m,n}^{\omega,b}(x_m, x_n). \quad (3.3)$$

Moreover, the minimizer is unique if $\psi(x) = K_{m,n}^{\omega,b} \varphi(\cdot)$ is differentiable at x , and in this case $v_n = \nabla \psi(x) + b$.

(2) Suppose $\psi_\omega^-(x, n)$ is the stationary solution. Then at every $x \in \mathbb{T}^d$ and $n \in \mathbb{Z}$, there exists a backward minimizer $(x_j, v_j)_{j=-\infty}^n$ such that $x_n = x$

$$K_{m,n}^{\omega,b} \psi_\omega^-(x_n, n) = \psi_\omega^-(x_m, m) + A_{m,n}^{\omega,b}(x_m, x_n), \quad m < n.$$

Moreover, the minimizer is unique if $\psi_\omega^-(\cdot, n)$ is differentiable at x , and in this case $v_n = \nabla \psi_\omega^-(x, n) + b$.

In case (1) we call $(x_j, v_j)_{j=m}^n$ a minimizer for $K_{m,n}^{\omega,b} \varphi(x_0)$, and in case (2) the orbit $(x_j, v_j)_{j=-\infty}^n$ is called a minimizer for $\psi^-(x_n, n)$.

The forward minimizer is linked to the forward operator $\check{K}_{s,t}^{\omega,b}$, defined as

$$\check{K}_{s,t}^{\omega,b} \varphi(x) = \sup_{y \in \mathbb{T}^d} \left\{ \varphi(y) - A_{s,t}^{\omega,b}(x, y) \right\}.$$

Analog of Proposition 2.1 and Lemma 3.1 hold, which we summarize below.

- For every $b \in \mathbb{R}^d$, almost surely, there exists a unique Lipschitz function $\psi_\omega^+(x, m)$, $m \in \mathbb{Z}$, such that

$$\check{K}_{m,n}^{\omega,b} \psi^+(x, n) = \psi^+(x, m), \quad m < n.$$

- For each $\check{K}_{m,n}^{\omega,b} \varphi(x)$ there exists a minimizer $(x_j, v_j)_{j=m}^n$ such that $x_m = x$ and

$$\check{K}_{m,n}^{\omega,b} \varphi(x_m) = \varphi(x_n) - A_{m,n}^{\omega,b}(x_m, x_n).$$

When $\psi = \check{K}_{m,n}^{\omega,b} \varphi$ is differentiable at x we have $v_m = \nabla \psi(x) + b$.

- For each $x \in \mathbb{T}^d$, and $m \in \mathbb{Z}$, there exists a forward minimizer $(x_j, v_j)_{j=m}^\infty$ such that $x_m = x$,

$$\check{K}_{m,n}^{\omega,b} \psi_\omega^+(x_m, m) = \psi_\omega^+(x_n, n) - A_{m,n}^{\omega,b}(x_m, x_n), \quad m < n,$$

and $v_m = \nabla \psi_\omega^+(x, m)$ if $\psi^+(\cdot, m)$ is differentiable at x .

The global minimizer is characterized by both ψ_ω^- and ψ_ω^+ .

Proposition 3.2. *Assume that Assumption 4 holds, and one of Assumptions 1 and 2 holds. Assume in addition, the map (2.4) is an embedding. Then for $b \in \mathbb{R}^d$, almost every ω , there exists a unique global minimizer $(x_j^\omega, v_j^\omega)_{j \in \mathbb{Z}}$. For each $j \in \mathbb{Z}$, x_j^ω is the unique $x \in \mathbb{T}^d$ reaching the minimum in*

$$\min_x \{ \psi_\omega^-(x, j) - \psi_\omega^+(x, j) \}. \quad (3.4)$$

Moreover, $\psi_\omega^\pm(\cdot, j)$ are both differentiable at x_j , and $v_j = \nabla \psi_\omega^-(x, j) + b = \nabla \psi_\omega^+(x, j) + b$.

The function

$$Q_\omega^\infty(x, j) := \psi_\omega^-(x, j) - \psi_\omega^+(x, j) \quad (3.5)$$

will serve an important purpose for the discussions below.

The random potentials F_j^ω are generated by a stationary random process, so there exists a measure preserving transformation θ on the probability space Ω satisfying

$$F_{n+m}^\omega(x) = F_n^{\theta^m \omega}(x). \quad (3.6)$$

The family of maps Φ_j^ω then defines a non-random transformation

$$\hat{\Phi}(x, v, \omega) = (\Phi_0(x, v), \theta\omega)$$

on the space $\mathbb{T}^d \times \mathbb{R}^d \times \Omega$. Then from Proposition 3.2,

$$(x_0^{\theta\omega}, v_0^{\theta\omega}) = \Phi_0^\omega(x_0^\omega, v_0^\omega)$$

and the probability measure

$$\nu(d(x, v), d\omega) = \delta_{(x_0^\omega, v_0^\omega)} P(d\omega)$$

is invariant and ergodic under $\hat{\Phi}$. The map $D\Phi_0^\omega : \mathbb{T}^d \times \mathbb{R}^d \times \Omega \rightarrow Sp(d)$, where $Sp(d)$ is the group of all $2d \times 2d$ symplectic matrices, defines a cocycle over $\hat{\Phi}$. Under Assumption 5, its Lyapunov exponents $\lambda_1(\nu), \dots, \lambda_{2d}(\nu)$ are well defined, and due to symplecticity, we have

$$\lambda_1(\nu) \leq \dots \leq \lambda_d(\nu) \leq 0 \leq \lambda_{d+1}(\nu) \leq \dots \leq \lambda_{2d}(\nu),$$

and $\lambda_i = -\lambda_{2d-i+1}$.

There is a close relation between the non-degeneracy of the variational problem (3.4), and non-vanishing of the Lyapunov exponents for the associated cocycle.

Proposition 3.3 ([8], Proposition 3.10). *Assume that assumption 5 and one of assumptions 1 or 2 holds. Assume in addition that the map (2.4) is an embedding. Then for all $b \in \mathbb{R}^d$, for a.e. ω , the following hold.*

- (1) *There exist $C(F, \rho), R(F, \rho) > 0$ depending only on F_1, \dots, F_M in (2.4) and the density ρ of ξ_j , and a positive random variable $a(\omega) > 0$ such that*

$$Q_\omega^\infty(x, 0) - Q_\omega^\infty(x_0^\omega, 0) \geq a(\omega) \|x - x_0^\omega\|^2, \quad \|x - x_0^\omega\| < R(F, \rho), \quad (3.7)$$

with

$$\mathbb{E}(a(\omega)^{-\frac{1}{2}}) < C(F, \rho). \quad (3.8)$$

- (2) *The Lyapunov exponents of ν satisfy*

$$\lambda_d(\nu) < 0 < \lambda_{d+1}(\nu).$$

The second conclusion of Proposition 3.3 implies the orbit (x_j^ω, v_j^ω) for the sequence of maps Φ_j^ω is non-uniformly hyperbolic. In particular, it follows that there exists local unstable and stable manifolds. It is shown in [8] that the graph of the gradient of the viscosity solutions locally coincide with the unstable and stable manifolds.

Proposition 3.4 ([8], Theorem 6.1). *Under the same assumptions as Proposition 3.3, for each $\epsilon > 0$, there exists positive random variables $r(\omega) > 0$, $C(\omega) > 1$, such that the following hold almost surely.*

- (1) *There exists C^1 embedded submanifolds $W^u(x_0^\omega, v_0^\omega)$ and $W^s(x_0^\omega, v_0^\omega)$, such that*

$$(x, \nabla \psi_\omega^-(x, 0) + b) \in W^u(x_0^\omega, v_0^\omega), \quad (x, \nabla \psi_\omega^+(x, 0) + b) \in W^s(x_0^\omega, v_0^\omega)$$

for all $\|x - x_0^\omega\| < r(\omega)$.

- (2) *For every $\|x - x_0^\omega\| < r(\omega)$, let $(x_j^-, v_j^-)_{j \leq 0}$ and $(x_j^+, v_j^+)_{j \geq 0}$ be the backward and forward minimizers satisfying $x_0^\pm = x$. Then*

$$\|(x_j^\omega, v_j^\omega) - (x_j^-, v_j^-)\| \leq C(\omega) e^{-\lambda' |j|}, \quad j \leq 0,$$

$$\|(x_j^\omega, v_j^\omega) - (x_j^+, v_j^+)\| \leq C(\omega) e^{-\lambda' |j|}, \quad j \geq 0,$$

where $\lambda' = \lambda_{d+1}(\nu) - \epsilon$.

Remark. In Lemma 4.9 we will show that the random variables $r(\omega), C(\omega)$ in item (2) can be chosen to satisfy an additional *tempered* property.

4. PROPERTIES OF THE VISCOSITY SOLUTIONS

4.1. Semi-concavity. Given $C > 0$, we say that a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is C semi-concave if for any $x \in \mathbb{R}^d$, there exists a linear form $l_x : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$f(y) - f(x) \leq l_x(y - x) + C\|y - x\|^2, \quad y \in \mathbb{R}^d.$$

A function $\varphi : \mathbb{T}^d \rightarrow \mathbb{R}$ is called C semi-concave if it is C semi-concave as a function lifted to \mathbb{R}^d . The linear form l_x is called a subdifferential at x . If φ is differentiable at $x \in \mathbb{T}^d$, then the subdifferential l_x is unique and is equal to $d\varphi(x)$. A semi-concave function is Lipschitz.

Lemma 4.1 ([5], Proposition 4.7.3). *If φ is continuous and C semi-concave on \mathbb{T}^d , then φ is $2C\sqrt{d}$ -Lipschitz.*

Lemma 4.2 ([5]). *Suppose both φ_1 and $-\check{\varphi}_2$ is C semi-concave, then over the set $\arg \min_x \{\varphi_1(x) - \check{\varphi}_2(x)\}$, $\varphi_1, \check{\varphi}_2$ are differentiable, $\nabla \varphi_1(x) = \nabla \varphi_2(x)$, and $\nabla \varphi_1(x)$ is $6C$ -Lipshitz over the set $\arg \min_x \{\varphi_1(x) - \check{\varphi}_2(x)\}$.*

Let $K_0(\omega) = \|F_0^\omega\|_{C^{2+\alpha}} + 1$ and $K(\omega) = 2\sqrt{d}(K_0(\omega) + 1)$. The action function $A_{m,n}^{\omega,b}$ has the following properties.

- Lemma 4.3** ([8], Lemma 3.2). (1) *The function $A_{m,n}^{\omega,b}(x, x')$ is 1-semi-concave in the second component, and is $K(\theta^m \omega)$ -semi-concave in the first component. Here $\theta : \Omega \rightarrow \Omega$ is the time-shift, see (3.6).*
- (2) *For any $\varphi \in C(\mathbb{T}^d)$, and $m < n \in \mathbb{Z}$, the function $K_{m,n}^{\omega,b}\varphi(x)$ is 1 semi-concave, and $-\check{K}_{m,n}^{\omega,b}\varphi(x)$ is $K_0(\theta^m \omega)$ semi-concave. Either function, as well as the sum of the two functions, are $K(\theta^m \omega)$ Lipshitz.*
- (3) *For $n \in \mathbb{Z}$, the functions $\psi_\omega^-(\cdot, n)$ is 1 semi-concave, and $-\psi_\omega^+(\cdot, n)$ is $K_0(\theta^n \omega)$ semi-concave, Either function, as well as the sum of the two functions, are $K(\theta^n \omega)$ Lipshitz.*

We first state two lemmas concerning the properties of the Lax-Oleinik semigroup, the goal is to obtain Lemma 4.7, which is a version of Mather's graph theorem ([10]).

Lemma 4.4. *For any $x \in \mathbb{T}^d$, $m < n$, $\varphi \in C(\mathbb{T}^d)$, we have*

$$\check{K}_{m,n}^{\omega,b} \left(K_{m,n}^{\omega,b} \varphi \right) (x) \leq \varphi(x)$$

Proof. For any $x, y \in \mathbb{T}^d$,

$$K_{m,n}^{\omega,b} \varphi(y) \leq \varphi(x) + A_{m,n}^{\omega,b}(x, y),$$

then

$$\check{K}_{m,n}^{\omega,b} \left(K_{m,n}^{\omega,b} \varphi \right) (x) = \max_{y \in \mathbb{T}^d} \left\{ K_{m,n}^{\omega,b}(y) - A_{m,n}^{\omega,b}(x, y) \right\} \leq \varphi(x).$$

□

Lemma 4.5. *Suppose $m < n$. Let $(x_j, v_j)_{j=m}^n$ be a minimizer for $K_{m,n}^{\omega,b}\varphi$ in the sense of (3.3). Then for each $m \leq j \leq n$, we have*

$$K_{m,j}^{\omega,b}\varphi(x) \geq \check{K}_{j,n}^{\omega,b} \left(K_{m,n}^{\omega,b} \varphi \right) (x), \quad \forall x \in \mathbb{T}^d,$$

and

$$K_{m,j}^{\omega,b}\varphi(x_j) = \check{K}_{j,n}^{\omega,b} \left(K_{m,n}^{\omega,b} \varphi \right) (x_j).$$

Proof. By definition $K_{nn}^{\omega,b}\varphi = \varphi$, so the case $j = n$ is trivial. For $m \leq j < n$, we have

$$K_{m,n}^{\omega,b}\varphi(x_n) = K_{m,j}^{\omega,b}\varphi(x_j) + A_{j,n}^{\omega,b}(x_j, x_n).$$

Then

$$K_{m,j}^{\omega,b}\varphi(x_j) = K_{m,n}^{\omega,b}\varphi(x_n) - A_{m,n}^{\omega,b}(x_j, x_n) \leq \check{K}_{j,n}^{\omega,b} \left(K_{m,n}^{\omega,b} \varphi \right) (x_j).$$

On the other hand, apply Lemma 4.4 to $K_{m,j}^{\omega,b}\varphi$ and $j < n$ yields

$$\check{K}_{j,n}^{\omega,b} \left(K_{m,n}^{\omega,b} \varphi \right) (x) = \check{K}_{j,n}^{\omega,b} \left(K_{j,n}^{\omega,b} K_{m,j}^{\omega,b} \varphi \right) (x) \leq K_{m,j}^{\omega,b}\varphi(x).$$

The lemma follows. □

As a result, A minimizer of a backward solution is also a minimizer of the forward-backward solution.

Corollary 4.6. *Let $(x_j, v_j)_{j=m}^n$ be a minimizer for $K_{m,n}^{\omega,b}\varphi$, then it is also a minimizer for $\check{K}_{m,n}^{\omega,b}(K_{m,n}^{\omega,b}\varphi)$.*

Proof. Using the calculations in the proof of Lemma 4.5, we get

$$\check{K}_{j,n}^{\omega,b}(K_{m,n}^{\omega,b}\varphi)(x_j) = K_{m,j}^{\omega,b}\varphi(x_j) = K_{m,n}^{\omega,b}\varphi(x_n) - A_{j,n}^{\omega,b}(x_j, x_n)$$

for all $m \leq j \leq n$. The corollary follows. \square

The following lemma provides a Lipschitz estimate for the velocity of the minimizer in the interior of the time interval.

Lemma 4.7. *Suppose $m < n$ with $n - m \geq 2$. Let $(x_j, v_j)_{j=m}^n$ and $(y_j, \eta_j)_{j=m}^n$ be two minimizers for $K_{m,n}^{\omega,b}\varphi$ in the sense of (3.3). Then for all $m < j < n$, we have*

$$\|v_j - \eta_j\| \leq K(\theta^j \omega) \|x_j - y_j\|.$$

The same conclusion hold, if (x_j, v_j) and (y_j, η_j) are minimizers for $\check{K}_{m,n}^{\omega,b}\varphi(x)$.

Proof. We apply Lemma 4.5 to (x_j, v_j) and (y_j, η_j) . Denote $\psi_1 = K_{m,j}^{\omega,j}\varphi$ and $\psi_2 = \check{K}_{j,n}^{\omega,b}(K_{m,n}^{\omega,b}\varphi)$, since $j - m, n - j \geq 1$, ψ_1 is 1 semi-concave, $-\psi_2$ is $K(\theta^j \omega)$ semi-concave. Since $x_j, y_j \in \arg \min_x \{\psi_1(x) - \psi_2(x)\}$, Lemma 4.2 and Lemma 3.1 implies

$$\|v_j - \eta_j\| = \|\nabla \psi_1(x_j) - \nabla \psi_1(y_j)\| \leq K(\theta^j \omega) \|x_j - y_j\|.$$

\square

4.2. Properties of the stationary solutions. Recall that

$$Q_\omega^\infty(x, n) = \psi_\omega^-(x, n) - \psi_\omega^+(x, n),$$

which takes its minimum at the global minimizer x_n^ω . To simplify notations, we will drop the subscript ω from these functions when there is no confusion.

This function Q^∞ is very useful, as it can be used to measure the distance to the global minimizer. For all $\|y - x_0^\omega\| < r(F)$, we have

$$a(\omega) \|y - x_0^\omega\|^2 \leq Q^\infty(x, 0) - Q^\infty(x_0^\omega, 0) \leq K(\omega) \|y - x_0^\omega\|^2. \quad (4.1)$$

Moreover, Q^ω is a Lyapunov function for infinite backward minimizers. Namely, if $(y_0, \eta_0) = (y_0, \nabla \psi_\omega^-(y_0, 0))$ is a backward minimizer, then for any $j < k \leq 0$, we have

$$Q^\infty(y_j, j) - Q^\infty(x_j^\omega, j) \leq Q^\infty(y_k, k) - Q^\infty(x_k^\omega, k). \quad (4.2)$$

(See [8], Lemma 7.2)

Let us also recall, for any $\lambda' < \lambda$, there exists functions $r(\omega)$, $C(\omega) > 0$ such that for all backward minimizers $(y_n, \eta_n)_{n \leq 0}$ such that

$$\|y_n - x_n^\omega\| \leq C(\omega) \exp(-\lambda' |n|), \quad \text{for } \|y_0 - x_0^\omega\| < r(\omega) \text{ and } n \leq 0. \quad (4.3)$$

We will also use a process in non-uniform hyperbolicity known as tempering.

Lemma 4.8 ([1], Lemma 3.5.7). *Let $g(\omega) > 1$ be a random variable satisfying $\mathbb{E}(\log g(\omega)) < \infty$, then for any $\epsilon > 0$, there exists $g^\epsilon(\omega) > g(\omega)$ such that*

$$e^{-\epsilon} \leq \frac{g^\epsilon(\omega)}{g^\epsilon(\theta\omega)} \leq e^\epsilon. \quad (4.4)$$

Let's call a random variable $g(\omega) > 0$ tempered if for any $\epsilon > 0$, both g, g^{-1} admits an upper bound satisfying (4.4). Products and inverses of tempered random variables are still tempered.

The random variables a, K in (4.1) and r, C in (4.3) are tempered.

Lemma 4.9. *For any $\epsilon > 0$, there exists random variables*

$$a^\epsilon(\omega) < a(\omega), \quad r^\epsilon(\omega) < r(\omega), \quad K^\epsilon(\omega) > K(\omega), \quad C^\epsilon(\omega) > C(\omega)$$

such that

$$e^{-\epsilon} \leq \frac{a^\epsilon(\omega)}{a^\epsilon(\theta\omega)}, \frac{r^\epsilon(\omega)}{r^\epsilon(\theta\omega)}, \frac{K^\epsilon(\omega)}{K^\epsilon(\theta\omega)}, \frac{C^\epsilon(\omega)}{C^\epsilon(\theta\omega)} \leq e^\epsilon.$$

Proof. Lemma 4.8 applies to $a(\omega)$ and $K(\omega)$ since $\mathbb{E}(\log a^{-1}), \mathbb{E}(\log K) < \infty$. The fact that $C(\omega)$ and $r(\omega)$ are tempered can be proven by adapting the proof Theorem 6.1 in [8]. We now explain the adaptations required.

In [8], there exists local linear coordinates (s, u) with the formula $(y, \eta) = P_j(s, u)$ centered at the global minimizer (x_j^ω, v_j^ω) , with the estimates $\|DP_j\|, \|DP_j^{-1}\| \leq K(\theta^j\omega)a^{-\frac{1}{2}}(\theta^j\omega)$ (section 6 of [8]). Then it is shown that there exists a random variable $r_0(\omega)$ (called r in that paper) such that any orbit $(y, \eta) \in \{(y, \nabla\psi_\omega^-(y, 0))\} \cap \{\|s\|, \|u\| < r_0\}$ must be contained in the stable manifold. r is tempered because it is the product of tempered random variables. Indeed, the following explicit formula was given in the Proof of Theorem 6.1, section 7 of [8]:

$$\bar{r} = \tilde{C}^{-3}K^{-9}a^6\kappa^2\rho^2, \quad r_0 = \bar{r}(\theta^{-1}\omega)K^3(\theta^{-1}\omega)a(\theta^{-1}\omega)\kappa^{\frac{1}{2}}(\theta^{-1}\omega),$$

where K, a are the same as in this paper, and the fact that ρ, \tilde{C} are tempered is explained in Lemma 6.5 and Proposition 7.1 of [8]. We now convert to the variable (y, η) . Since the norm of the coordinate changes are tempered, there exists a tempered random variable $r_1(\omega)$ such that any orbit contained in

$$\{(y, \nabla\psi_\omega^-(y, 0))\} \cap \{\|(y, \eta) - (x_0^\omega, v_0^\omega)\| < r_1(\omega)\}$$

must be contained in the unstable manifold of (x_0^ω, v_0^ω) .

We now show the same conclusion holds on a neighborhood of the configuration space $\|y - x_0^\omega\| < r(\omega)$, with $r(\omega)$ tempered. Let $(y_0, \eta_0) = (y_0, \nabla\psi^-(y_0, 0))$ and let (y_{-1}, η_{-1}) be its backward image. According to Lemma 4.2, $\|\eta_{-1} - v_{-1}^\omega\| \leq K(\theta^{-1}\omega)\|y_{-1} - x_{-1}^\omega\|$, as a result

$$\|y_{-1} - x_{-1}^\omega\| < \frac{r_1}{1 + K}((\theta^{-1}\omega)) \text{ implies } \|(y_{-1}, \eta_{-1}) - (x_{-1}^\omega, v_{-1}^\omega)\| < r_1(\theta^{-1}\omega).$$

Finally, using (4.1) and (4.2), we have

$$\begin{aligned} \|y_{-1} - x_{-1}\| &\leq a^{\frac{1}{2}}(\theta^{-1}\omega) (Q^\infty(y_{-1}, -1) - Q^\infty(x_{-1}, -1)) \\ &\leq a^{\frac{1}{2}}(\theta^{-1}\omega) (Q^\infty(y_0, 0) - Q^\infty(x_0, 0)) \leq a^{\frac{1}{2}}(\theta^{-1}\omega) K^{\frac{1}{2}}(\omega) \|y_0 - x_0\|, \end{aligned}$$

We obtain that

$$\|y_0 - x_0^\omega\| < \frac{r_1}{(1+K)(a^{-\frac{1}{2}} \circ \theta^{-1})K^{\frac{1}{2}}} =: r$$

implies (y_1, η_1) is contained in the unstable manifold of $(x_{-1}^\omega, v_{-1}^\omega)$. r is tempered as it is products of tempered random variables.

The fact that an orbit on the stable manifold of a non-uniformly hyperbolic orbit converge at the rate $C(\omega)e^{-\lambda n}$, and that the coefficient $C(\omega)$ is tempered is a standard result in non-uniform hyperbolicity, see for example [1]. \square

We now use what we obtained to get an approximation for the stationary solutions.

Lemma 4.10. *There exists $C_1^\epsilon(\omega) > 0$, $e^{-\epsilon} \leq C_1(\omega)/C_1(\theta^\omega) \leq e^\epsilon$, such that for all $\|y - x_0^\omega\| < r(\omega)$, and $n < 0$, we have*

$$\|\psi^-(y, 0) - \psi^-(x_n, n) - A_{n,0}^{\omega,b}(x_n, y)\| \leq C_1(\omega)e^{-(\lambda' - \epsilon)|n|}.$$

We also have the forward version: for $n > 0$,

$$\|\psi^+(y, 0) - \psi^+(x_n, n) + A_{0,n}^{\omega,b}(y, x_n)\| \leq C_1(\omega)e^{-(\lambda' - \epsilon)|n|}.$$

Proof. We only prove the backward version. By definition,

$$\psi^-(y, 0) \leq \psi^-(x_n, n) + A_{n,0}^{\omega,b}(x_n, y).$$

On the other hand, let $(y_n, \eta_n)_{n \leq 0}$ be the minimizer for $\psi_\omega^-(\psi, 0)$, then

$$\begin{aligned} \psi^-(y, 0) &= \psi^-(y_n, n) + A_{n,0}^{\omega,b}(y_n, y) \\ &\geq \psi^-(x_n, n) + A_{n,0}^{\omega,b}(x_n, y) - K^\epsilon(\theta^n \omega) \|x_n - y_n\| \\ &\geq \psi^-(x_n, n) + A_{n,0}^{\omega,b}(x_n, y) - K^\epsilon(\omega) C^\epsilon(\omega) e^{-(\lambda' - \epsilon)|n|}, \end{aligned}$$

where C^ϵ is from Lemma 4.9. The lemma follows by repeating the calculation with $\epsilon/2$, and taking $C_1^\epsilon(\omega) = K^{\epsilon/2}(\omega) C^{\epsilon/2}(\omega)$. \square

5. REDUCING TO LOCAL CONVERGENCE

In this section we reduce the main theorem to its local version.

Proposition 5.1. *Under the same assumptions as Proposition 3.3, there exists $0 < \lambda < \lambda_{d+1}(\nu)$, positive random variables $0 < \tau(\omega) < r(\omega)$, $D_0(\omega) > 0$, and $N_0(\omega) > 0$ such that for all $N > N_0(\omega)$ and $\varphi \in C(\mathbb{T})$,*

$$\sup_{\varphi \in C(\mathbb{T})} \min_{C \in \mathbb{R}} \max_{\|y - x_0^\omega\| \leq \tau(\omega)} |K_{-N,0}^{\omega,b} \varphi(y) - \psi_\omega^-(y, 0) - C| \leq D_0(\omega) e^{-\lambda N}.$$

The proof of Proposition 5.1 is given in the next section. Next, we have a localization result, which says any minimizer for $K_{-N,M}^{\omega,b} \varphi$ or $\psi^-(\cdot, M)$ go through the neighborhood $\{\|x - x_0^\omega\| < \rho(\omega)\}$ at time $t = 0$, when N, M are large enough.

Proposition 5.2. *Under the same assumptions as Proposition 3.3, let $\tilde{\rho}(\omega) > 0$ be a positive random variable. Then there exists $M_0 = M_0(\omega) \in \mathbb{N}$ depending on $\tilde{\rho}(\omega), \omega$ such that the following hold.*

- (1) For any $N, M \geq M_0(\omega)$, let (y_n, η_n) , $-N \leq n \leq M$ be a (backward) minimizer for $K_{-N,M}^{\omega,b} \varphi(y_M)$. Then $\|y_0 - x_0^\omega\| < \tilde{\rho}(\omega)$.
- (2) (The forward version) For any $N, M \geq M_0(\omega)$, let (y_n, η_n) , $-N \leq n \leq M$ be a (forward) minimizer for $\tilde{K}_{-N,M}^{\omega,b} \varphi(y_{-N})$. Then $\|y_0 - x_0^\omega\| < \tilde{\rho}(\omega)$.

We prove Proposition 5.2 using the following lemma, stating that the Lax-Oleinik operators are weak contractions.

Lemma 5.3 ([6], Lemma 3). For any $\varphi_1, \tilde{\varphi}_2 \in C(\mathbb{T}^d)$, we have

$$\|K_{m,n}^{\omega,b} \varphi_1 - K_{m,n}^{\omega,b} \tilde{\varphi}_2\|_* \leq \|\varphi_1 - \tilde{\varphi}_2\|_*.$$

Proof of Proposition 5.2. We only prove item (1), as the (2) can be proven in the same way. We denote, for $-N \leq m \leq M$,

$$\psi^{-N,m}(x) = K_{-N,m}^{\omega,b} \varphi(x),$$

and let (y_n, η_n) , $-N \leq n \leq M$ be a minimizer for $K_{-N,M}^{\omega,b} \varphi$. For each $-N < n < M$, Lemma 4.4 and 4.5 implies

$$\psi^{-N,n}(x) - \tilde{K}_{n,M}^{\omega,b}(\psi^{-N,M})(x) \geq 0, \quad x \in \mathbb{T}^d,$$

$$\psi^{-N,n}(y_n) - \tilde{K}_{n,M}^{\omega,b}(\psi^{-N,M})(y_n) = 0,$$

hence

$$y_n \in \arg \min_x \left\{ \psi^{-N,n}(x) - \tilde{K}_{n,M}^{\omega,b}(\psi^{-N,M})(x) \right\}. \quad (5.1)$$

Recall that for $Q^\infty(x, 0) = \psi^-(x, 0) - \psi^+(x, 0)$, we have x_0^ω is the unique minimum for $Q^\infty(\cdot, 0)$. Define

$$\delta(\omega) = \min_{\|x - x_0^\omega\| \geq \tilde{\rho}(\omega)} Q^\infty(x) - Q^\infty(x_0^\omega).$$

By Proposition 2.1, we can choose $M_0(\omega)$ large enough such that for $N, M \geq M_0(\omega)$,

$$\|\psi^{-N,0} - \psi^-(\cdot, 0)\|_*, \|\tilde{K}_{0,M}^{\omega,b}(\psi^{-N,M}) - \psi^+(\cdot, 0)\|_* < \delta(\omega)/4.$$

As a result, there exists a constant $C \in \mathbb{R}$ such that

$$\left\| \left(\psi^{-N,0} - \tilde{K}_{0,M}^{\omega,b}(\psi^{-N,M}) \right) - Q^\infty(x, 0) - C \right\| < \delta(\omega)/2.$$

It follows that the minimum in (5.1) is never reached outside of $\{\|x - x_0^\omega\| < \rho(\omega)\}$. We obtain $\|y_0 - x_0^\omega\| < \tilde{\rho}(\omega)$. \square

We now prove our main theorem assuming Proposition 5.1.

Proof of Theorem 1. It suffices to prove the theorem for $n = 0$.

Let us apply Proposition 5.2 with $\tilde{\rho}(\omega) = \rho(\omega)$ from Proposition 5.1. Let (y_n, η_n) be a minimizer of $K_{-N,M}^{\omega,b} \varphi(y_M)$, and $(\tilde{y}, \tilde{\eta}_n)$ a minimizer for

$$\psi^-(y_N, N) = \left(K_{-M,N}^{\omega,b} \psi^-(\cdot, -M) \right)(y_N)$$

with $y_M = \tilde{y}_M$, and $M = M_0(\omega)$. According to Proposition 5.1, there exists $C(N, \omega) \in \mathbb{R}$ such that for all $\|y - x_0^\omega\| < \rho(\omega)$,

$$\left| K_{-N,0}^{\omega,b} \varphi(y) - \psi_\omega^-(y, 0) - C(N, \omega) \right| \leq D_0(\omega) e^{-\lambda N}. \quad (5.2)$$

Then

$$\begin{aligned} K_{-N,M}^{\omega,b} \varphi(y_M) &= K_{-N,0}^{\omega,b} \varphi(y_M) + A_{0,M}^{\omega,b}(y_0, y_M) \\ &= (K_{-N,0}^{\omega,b} \varphi(y_0) - \psi_\omega^-(y_0, 0)) + \psi_\omega^-(y_0, 0) + A_{0,M}^{\omega,b}(y_0, y_M) \\ &\geq C(N, \omega) + \psi^-(y_M, M) - D_0(\omega) e^{-\lambda N}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi^-(\tilde{y}_M, M) &= \psi^-(\tilde{y}_0, 0) + A_{0,M}^{\omega,b}(\tilde{y}_0, \tilde{y}_M) \\ &= (\psi^-(\tilde{y}_0, 0) - K_{-N,0}^{\omega,b} \varphi(\tilde{y})) + K_{-N,0}^{\omega,b} \varphi(\tilde{y}_0, \tilde{y}_M) + A_{0,M}^{\omega,b}(\tilde{y}_0, \tilde{y}_M) \\ &\leq -C(N, \omega) + K_{-N,M}^{\omega,b} \varphi(\tilde{y}_M) + D_0(\omega) e^{-\lambda N}. \end{aligned}$$

Using $y_M = \tilde{y}_M$, combine both estimates, and take supremum over all y_M , we get

$$\sup_{y \in \mathbb{T}^d} \left| K_{-N,M}^{\omega,b} \varphi(y) - \psi^-(y, M) - C(N, \omega) \right| \leq D_0(\omega) e^{-\lambda N}.$$

In order to shift the end time to $t = 0$, let denote $E_Q = \{\omega \in \Omega : M_0(\omega) \leq Q\}$ for $Q \in \mathbb{N}$. Fix some Q such that $P(E_Q) > 0$, then by ergodicity, almost surely, $M(\theta^k \omega) \leq Q$ for infinitely many k . Let us define $k(\omega)$ as the largest $k < -Q$ such that $M_0(\theta^k \omega) = Q$, then

$$\begin{aligned} \left\| K_{-N,0}^{\omega,b} \varphi - \psi_\omega^-(\cdot, 0) \right\| &= \left\| K_{-N-k,k}^{\theta^k \omega,b} \varphi - \psi_{\theta^k \omega}^-(\cdot, -k) \right\| \\ &\leq D_0(\theta^k \omega) e^{-\lambda(N-k)} = D_0(\theta^k \omega) e^{\lambda k(\omega)} e^{-\lambda N} \end{aligned}$$

provided $N \geq \max\{Q, N_0(\theta^k \omega)\} - k$. By reducing λ and taking N larger we can absorb the constant $D_0(\theta^k \omega) e^{\lambda k(\omega)}$. \square

6. LOCAL CONVERGENCE: LOCALIZATION AND UPGRADE

In this section we prove Proposition 5.1 (local convergence) using Proposition 6.1 and a consecutive upgrade scheme. It is useful to have the following definition for book keeping.

Definition. Given $\delta > 0$, $N \in \mathbb{N}$, let $(y_n, \eta_n)_{n=-N}^0$ be a minimizer for $\psi_\omega^N(\cdot, 0) = K_{-N,0}^{\omega,b} \varphi(\cdot)$, we say the orbit satisfies the (backward) (φ, δ, N) approximation property if for every $-N/3 \leq n \leq 0$ such that $\|y_n - x_n^\omega\| < r(\theta^n \omega)$, we have

$$\left| (\psi_\omega^N(y_n, n) - \psi_\omega^N(x_n^\omega, n)) - (\psi_\omega^-(y_n, n) - \psi_\omega^-(x_n, n)) \right| < \delta.$$

We denote this condition $\mathbf{AP}^-(\omega, \varphi, \delta, N)$.

The following proposition is our main technical result, which says the approximation property allows us to estimate how close a backward minimizer is to the global minimizer:

Proposition 6.1. *Let $0 < \epsilon < \lambda'/6$, there exists random variables $0 < \rho(\omega) < r(\omega)$, $N_1(\omega) > 0$ depending on ϵ , such that if a ψ_ω^N backward minimizer $(y_n, \eta_n)_{n=-N}^0$ satisfies the $\mathbf{AP}^-(\omega, \varphi, \delta, N)$ condition, with ¹*

$$\|y_0 - x_0^\omega\| < \rho(\omega), \quad N > N_1(\omega), \quad \delta^{\frac{1}{8}} < \rho(\omega), \quad \delta \geq e^{-(\lambda' - 3\epsilon)N/3}, \quad (6.1)$$

then there exists $\max\{\frac{1}{8\epsilon} \log \delta, -\frac{N}{6}\} \leq k < 0$ such that

$$\|y_k - x_k^\omega\| < \max\left\{\delta^q, \quad e^{-(\lambda' - 3\epsilon)N/6}\right\}, \quad (6.2)$$

where $q = (\lambda' - 3\epsilon)/(8\epsilon)$.

The proof require a detailed analysis using hyperbolic theory, and is deferred to the next few sections. In this section we prove Proposition 5.1 assuming Proposition 6.1.

We need to use both the forward and backward dynamics.

Definition. *Given $\delta > 0$, $N \in \mathbb{N}$, let $(y_n, \eta_n)_{n=0}^N$ be a minimizer for $\check{\psi}_\omega^N(\cdot, 0) = \check{K}_{0,N}^{\omega,b} \varphi(\cdot)$, we say the orbit satisfies the (forward) (φ, δ, N) approximation property, if for every $0 \leq k \leq N/3$ such that $\|y_k - x_k^\omega\| < r(\theta^k \omega)$, we have*

$$\left| \left(\check{\psi}_\omega^N(y_k, k) - \check{\psi}_\omega^N(x_k^\omega, k) \right) - \left(\psi_\omega^+(y_k, k) - \psi_\omega^+(x_k, k) \right) \right| < \delta.$$

We denote this condition $\mathbf{AP}^+(\omega, \varphi, \delta, N)$.

We state a forward version of Proposition 6.1. The proof is the same.

Proposition 6.2. *There exists random variables $0 < \check{\rho}(\omega) < r(\omega)$ and $\check{N}_1(\omega) > 0$ such that if a $\check{\psi}_\omega^N$ backward minimizer $(y_n, \eta_n)_{n=0}^N$ satisfies the $\mathbf{AP}^+(\omega, \varphi, \delta, N)$ condition, and in addition,*

$$\|y_0 - x_0^\omega\| < \check{\rho}(\omega), \quad N > \check{N}_1(\omega), \quad \delta^{\frac{1}{8}} < \check{\rho}(\omega), \quad \delta \geq \check{C}_2(\omega) e^{-(\lambda' - 2\epsilon)N/3}, \quad (6.3)$$

then there exists $0 < k \leq \max\{-\frac{1}{8\epsilon} \log \delta, \frac{N}{6}\}$ such that

$$\|y_k - x_k^\omega\| < \max\left\{\delta^q, \quad e^{-(\lambda' - 3\epsilon)N/6}\right\}, \quad (6.4)$$

where $q = (\lambda' - 3\epsilon)/(8\epsilon)$.

The main idea for the proof of Proposition 5.1 is to use both the forward and backward dynamics to repeatedly upgrade the estimates. If we have a \mathbf{AP}^- condition, Proposition 6.1 implies upgraded localization of *backward* minimizers at a earlier time. This can be applied to get a better approximation of the *forward* solution for the later time, obtaining an improved \mathbf{AP}^+ condition. We then reverse time and repeat. However, due to technical reasons, we can only apply this process on a sub-interval called a good interval.

¹The exponential on lower bound in δ in (6.1) is for simpler calculation and comes with no loss of generality. Indeed, if δ is exponentially small we already have the conclusion of Proposition 5.1.

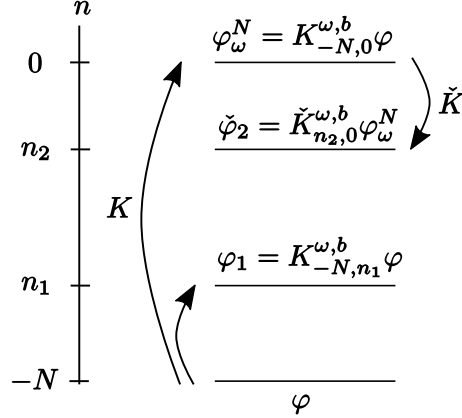


FIGURE 1. Notations for forward-backward iteration

Definition. For $\beta > 0$, we say $j \in [-N, 0]$ is a backward β -good time if for every $\varphi \in C(\mathbb{T}^d)$ and every $K_{-N,0}^{\omega,b} \varphi$ minimizer $(y_n, \eta_n)_{n=-N}^0$, we have

$$\|y_j - x_j^\omega\| < \beta < \rho(\theta^j \omega), \quad K^\epsilon(\theta^j \omega) < \beta^{-1}, \quad N_1(\omega) < \beta^{-1}$$

where $\rho(\omega)$ is in Proposition 6.1. Define forward β -good time similarly, by using forward minimizers and $\check{\rho}$ from Proposition 6.2. An interval $[n_1, n_2]$ is good if n_2 is backward good and n_1 is forward good.

Write $\varphi_\omega^N = K_{-N,0}^{\omega,b} \varphi$, note that by Corollary 4.6, if (y_n, η_n) is a minimizer for $K_{-N,0}^{\omega,b} \varphi$, then it is also a minimizer for $\check{K}_{-N,0}^{\omega,b} \varphi_\omega^N = \check{K}_{-N,0}^{\omega,b} K_{-N,0}^{\omega,b} \varphi$. For a β -good interval, denote

$$\begin{aligned} \omega_1 &= \theta^{n_1} \omega, & \varphi_1 &= K_{-N,n_1}^{\omega,b} \varphi, \\ \omega_2 &= \theta^{n_2} \omega, & \check{\varphi}_2 &= \check{K}_{n_2,0}^{\omega,b} \varphi_\omega^N = \check{K}_{n_2,0}^{\omega,b} K_{-N,0}^{\omega,b} \varphi. \end{aligned} \tag{6.5}$$

See Figure 1 for a visualization of the notations.

We now describe the upgrade lemma:

Lemma 6.3. For $0 < \epsilon < \lambda'/12$ and $\beta > 0$, there exists $N_2(\omega) > 0$ such that if

$$N > N_2(\omega), \quad \delta^{\frac{1}{8}} < \beta/2, \quad \delta \geq e^{-(\lambda'-3\epsilon)N/3},$$

and if $[-5N/9, -4N/9] \subset [n_1, n_2] \subset [-2N/3, -N/3]$ is a β -good time interval with regards to $[-N, 0]$, we have:

- (1) For $\omega_1, \omega_2, \varphi_1, \check{\varphi}_2$ defined in (6.5) and $\bar{N} = n_2 - n_1$, if

$$(y_{j+n_2}, \eta_{j+n_2})_{j=-\bar{N}}^0 \text{ satisfies } \mathbf{AP}^-(\omega_2, \varphi_1, \delta, \bar{N}) \text{ condition,}$$

then the orbit

$$(y_{j+n_1}, \eta_{j+n_1})_{j=0}^{\bar{N}} \text{ satisfies } \mathbf{AP}^+(\omega_1, \check{\varphi}_2, \delta_1, \bar{N}) \text{ condition,}$$

where

$$\delta_1 = \max \left\{ \delta^{q-\frac{1}{4}}, e^{-(\lambda'-6\epsilon)N/54} \right\}. \tag{6.6}$$

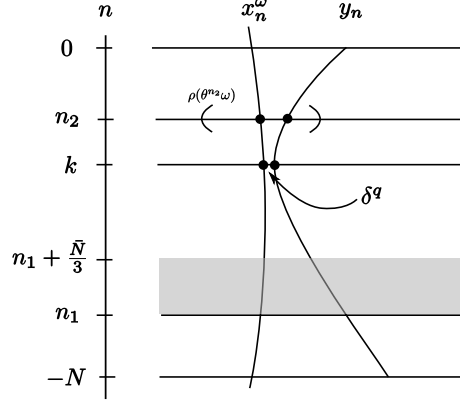


FIGURE 2. Proof of Lemma 6.3, part (1): Use early localization to estimate forward solutions on the gray interval

(2) If

$(y_{j+n_1}, \eta_{j+n_1})_{j=0}^{\bar{N}}$ satisfies $\mathbf{AP}^+(\omega_1, \check{\varphi}_2, \delta, \bar{N})$ condition,

then

$(y_{j+n_2}, \eta_{j+n_2})_{j=-\bar{N}}^0$ satisfies $\mathbf{AP}^-(\omega_2, \varphi_1, \delta_1, \bar{N})$ condition,

with δ_1 given by (6.6).

The reason to require the good time interval to lie in $[-2N/3, -N/3]$ is to apply the following lemma, which says the global minimizer x_n^ω is almost a minimizer for finite time solution in the middle of the time interval:

Lemma 6.4. *There exists $N_3(\omega) > 0$ such that if $N > N_3(\omega)$, the following holds almost surely, for arbitrary $\varphi \in C(\mathbb{T}^d)$ and $-2N/3 \leq j < k \leq -N/3$:*

$$\begin{aligned} \left| K_{-N,j}^{\omega,b} \varphi(x_j^\omega) - K_{-N,k}^{\omega,b} \varphi(x_k^\omega) + A_{j,k}^{\omega,b}(x_j^\omega, x_k^\omega) \right| &< e^{-(\lambda' - 3\epsilon)N/6}, \\ \left| \check{K}_{j,0}^{\omega,b} \varphi(x_j^\omega) - \check{K}_{k,0}^{\omega,b} \varphi(x_k^\omega) + A_{j,k}^{\omega,b}(x_j^\omega, x_k^\omega) \right| &< e^{-(\lambda' - 3\epsilon)N/6}. \end{aligned}$$

The proof of the lemma is deferred to section 7.1.

Proof of Lemma 6.3. Refer to Figure 2 for an illustration of the strategy of the proof.

Since n_2 is a backward good time and $\delta^{\frac{1}{8}} < \beta < \rho(\theta^{n_2}\omega) = \rho(\omega_2)$, condition (6.1) holds for the orbit $(y_{j+n_2}, \eta_{j+n_2})_{j=-\bar{N}}^0$ at the shifted time ω_2 , the initial condition φ_1 , and interval size \bar{N} .

Therefore by Proposition 6.1, there exists $0 > -n_2 \geq \max\{-\frac{\bar{N}}{6}, \frac{1}{8\epsilon} \log \delta\}$, such that

$$\|y_k - x_k^\omega\| < \max\{\delta^q, e^{-(\lambda' - 3\epsilon)\bar{N}/6}\}.$$

Also, $\bar{N} \geq N/9 \geq \beta^{-1} \geq N_1(\omega_2)$ provided $N \geq 9\beta^{-1}$. Consider $n_1 \leq j \leq \bar{N}/3 + n_1 < k$, such that $\|y_j - x_j^\omega\| < r(\theta^j\omega)$. By Corollary 4.6,

$$\check{K}_{j,0}^{\omega} \varphi_\omega^N(y_j) = \check{K}_{k,0}^{\omega} \varphi_\omega^N(y_k) - A_{k,j}(y_k, y_j),$$

then

$$\begin{aligned}
& \left| \tilde{K}_{j,0} \varphi_\omega^N(y_j) - \tilde{K}_{k,0} \varphi_\omega^N(x_k^\omega) + A_{k,j}(x_k^\omega, y_j) \right| \\
& \leq \left| \tilde{K}_{k,0} \varphi_\omega^N(y_k) - \tilde{K}_{k,0} \varphi_\omega^N(x_k^\omega) \right| + |A_{k,j}(y_k, y_j) - A_{k,j}(x_k^\omega, y_j)| \\
& \leq K^\epsilon(\theta^k \omega) \|x_k^\omega - y_k\| \leq K^\epsilon(\theta^{n_2} \omega) e^{\epsilon|k-n_2|} \max \left\{ \delta^q, e^{-(\lambda'-3\epsilon)\bar{N}/6} \right\} \\
& \leq \beta^{-1} \max \left\{ \delta^{q-\frac{1}{8}}, e^{-(\lambda'-4\epsilon)\bar{N}/6} \right\} \leq \beta^{-1} \max \left\{ \delta^{q-\frac{1}{8}}, e^{-(\lambda'-4\epsilon)N/54} \right\}.
\end{aligned} \tag{6.7}$$

using Lemma 4.7, and the estimates $e^{\epsilon|k-n_2|} \leq \delta^{-\frac{1}{8}}$, $e^{\epsilon|k-n_2|} \leq e^{\epsilon\bar{N}/6}$ and $\bar{N} \geq N/9$.

On the other hand, the dual version of Lemma 4.10 implies

$$\begin{aligned}
& \left| \psi_\omega^+(y_j, j) - \psi_\omega^+(x_k^\omega, k) + A_{k,j}(x_k^\omega, x_j^\omega) \right| < C_1^\epsilon(\theta^j \omega) e^{-(\lambda'-\epsilon)(k-j)} \\
& \leq C_1^\epsilon(\omega) e^{\epsilon|j|} e^{-(\lambda'-\epsilon)\bar{N}/2} \leq C_1^\epsilon(\omega) e^{2\epsilon N/3} e^{-(\lambda'-\epsilon)N/18} = C_1^\epsilon(\omega) e^{-(\lambda'-13\epsilon)N/18}.
\end{aligned}$$

Combine the estimates, we get

$$\begin{aligned}
& \left| \left(\tilde{K}_{j,0}^{\omega,b} \varphi_\omega^N(y_j) - \tilde{K}_{k,0}^{\omega,b} \varphi_\omega^N(x_k^\omega) \right) - (\psi_\omega^+(y_j, j) - \psi_\omega^+(x_k^\omega, k)) \right| \\
& \leq \beta^{-1} \max \left\{ \delta^{q-\frac{1}{8}}, e^{-(\lambda'-4\epsilon)N/54} \right\} + C_1^\epsilon(\omega) e^{-(\lambda'-13\epsilon)N/18}.
\end{aligned}$$

We now apply Lemma 6.4 to $\varphi = \varphi_\omega^N$, to replace the index k with j :

$$\begin{aligned}
& \left| \left(\tilde{K}_{j,0}^{\omega,b} \varphi_\omega^N(y_j) - \tilde{K}_{j,0}^{\omega,b} \varphi_\omega^N(x_j^\omega) \right) - (\psi_\omega^+(y_j, j) - \psi_\omega^+(x_j^\omega, j)) \right| \\
& \leq \beta^{-1} \max \left\{ \delta^{q-\frac{1}{8}}, e^{-(\lambda'-4\epsilon)N/54} \right\} + C_1(\omega) e^{-(\lambda'-\epsilon)N/3} + e^{-(\lambda'-3\epsilon)N/6} \\
& \leq 2\beta^{-1} \max \left\{ \delta^{q-\frac{1}{8}}, e^{-(\lambda'-4\epsilon)N/54} \right\} < \max \left\{ \delta^{q-\frac{1}{4}}, e^{-(\lambda'-5\epsilon)N/54} \right\} = \delta_1,
\end{aligned} \tag{6.8}$$

where in the last inequality, we take $N_2(\omega)$ large enough so that $C_1(\omega) e^{-(\lambda'-\epsilon)N/3} + e^{-(\lambda'-3\epsilon)N/6} < e^{-(\lambda'-4\epsilon)N/54}$ and $2\beta^{-1} < e^{-\epsilon N/54}$, then we use $2\beta^{-1} < \delta^{-\frac{1}{8}}$.

Observe that by the standard semi-group property,

$$\tilde{K}_{j-n_1, \bar{N}}^{\omega_1, b} \tilde{\varphi}_2 = \tilde{K}_{j-n_1, \bar{N}}^{\theta^{n_1} \omega, b} \tilde{K}_{n_2, 0}^{\omega, b} \varphi_\omega^N = \tilde{K}_{j, n_2}^{\omega, b} \tilde{K}_{n_2, 0}^{\omega, b} \varphi_\omega^N = \tilde{K}_{j, 0}^{\omega, b} \varphi_\omega^N,$$

substitute into (6.8) we obtain $\mathbf{AP}^+(\omega_1, \tilde{\varphi}_2, \delta_1, \bar{N})$.

We now discuss case 2. Starting with the condition $\mathbf{AP}^+(\omega_1, \tilde{\varphi}_2, \delta, \bar{N})$, we obtain $0 \leq k_* - n_1 \leq \max \left\{ -\frac{1}{8\epsilon} \log \delta, \frac{N}{6} \right\}$ such that

$$\|y_{k_*} - x_{k_*}\| \leq \max \left\{ \delta^q, e^{-(\lambda'-3\epsilon)N/6} \right\}.$$

Then for all $k_* < n_2 - \bar{N}/3 \leq j \leq n_2$, if $\|y_j - x_j^\omega\| < r(\theta^j \omega)$, using the fact that (y_n, η_n) is a minimizer for $K_{-N, 0} \varphi$, similar to (6.7) we have

$$\left| K_{-N, j}^{\omega, b} \varphi(y_j) - K_{-N, k}^{\omega, b} \varphi(x_k^\omega) + A_{j, k}^{\omega, b}(x_k^\omega, y_j) \right| \leq \beta^{-1} \max \left\{ \delta^{q-\frac{1}{8}}, e^{-(\lambda'-4\epsilon)N/54} \right\},$$

and following the same strategy as before, we get

$$\left| \left(K_{-N, j}^{\omega, b} \varphi(y_j) - K_{-N, j}^{\omega, b} \varphi(x_j^\omega) \right) - (\psi^-(y_j, j) - \psi^-(x_j^\omega, j)) \right| < \delta_1.$$

Since

$$K_{-\bar{N}, j-n_2}^{\omega_2, b} \varphi_1 = K_{n_1, j}^{\omega, b} K_{-N, n_1} \varphi = K_{-N, j}^{\omega, b} \varphi,$$

we obtain $\mathbf{AP}^-(\omega_2, \varphi_1, \delta_1, \bar{N})$. \square

To carry out the upgrading procedure, we need to show that β -good time intervals exist.

Lemma 6.5. *There exists $\beta_0 > 0$ such that, for any $0 < \beta \leq \beta_0$, there exists $N_3(\omega) > 0$, and for all $N > N_3(\omega)$ there exists a β -regular time interval $[-5N/9, -4N/9] \subset [n_1, n_2] \subset [-2N/3, -N/3]$.*

Proof. Let $\beta_0 > 0$ be small enough that

$$P(\rho(\omega) > \beta_0, K^\epsilon(\omega) < \beta_0^{-1}, N_1(\omega) < \beta_0^{-1}) > \frac{17}{18}.$$

By Proposition 5.2, for any $0 < \beta \leq \beta_0$, there exists $M_0(\omega) > 0$ such that any minimizer $(y_n, \eta_n)_{n=-M}^N$ with $M, N > M_0(\omega)$ satisfies $\|y_0 - x_0^\omega\| < \beta$. We now choose $\beta_1 > 0$ small enough such that

$$P(\rho(\omega) > \beta, K^\epsilon(\omega) < \beta^{-1}, N_1(\omega) < \beta^{-1}, M_0(\omega) < \beta^{-1}) > \frac{8}{9}.$$

Then, there exists $N_3(\omega) > 0$ such that for all $N > N_3(\omega)$ the density of β -regular n in $[-N, 0]$ is larger than $\frac{8}{9}$. In particular, the interval $[-4N/9, -N/3]$ must contain a regular time n_2 . We impose $N_3(\omega) > 3\beta_1^{-1}$, then Proposition 5.2 implies for any $N > N_3(\omega)$, $\|y_{n_2} - x_{n_2}^\omega\| < \beta \leq \rho(\omega)$, therefore n_2 is a good time.

Apply the same argument, by possibly choosing a different $N_3(\omega)$, we can find a forward good time n_1 in $[-2N/3, -5N/9]$. \square

Proof of Proposition 5.1. By Lemma 6.5 there exists a β -good interval. We first show that if for $K_{-N, 0}^{\omega, b} \varphi$ backward minimizer $(y_n, \eta_n)_{n=-N}^0$, the condition

$$\mathbf{AP}^-(\omega_2, \varphi_1, e^{-\lambda_1 N}, \bar{N}) \quad (6.9)$$

holds for an explicitly defined λ_1 , then Proposition 5.1 follows. Indeed, we only need the estimate

$$|\psi_\omega^N(y_{n_2}, n_2) - \psi_\omega^-(y_{n_2}, n_2) - C(n_2, \omega, \varphi)| \leq e^{-\lambda_1 N},$$

where $C(n_2, \omega, \varphi) = \psi_\omega^N(x_{n_2}^\omega, n_2) - \psi_\omega^-(x_{n_2}^\omega, n_2)$.

On one hand,

$$\psi_\omega^N(y_0, 0) - \psi^-(y_0, 0) \geq \psi^N(y_{n_2}, n_2) - \psi^-(y_{n_2}, n_2) \geq C(n_2, \omega, \varphi) - e^{-\lambda_1 N},$$

on the other hand, by Lemma 4.10,

$$\begin{aligned} \psi_\omega^N(y_0, 0) - \psi^-(y_0, 0) &\leq \psi^N(x_{n_2}^\omega, n_2) - \psi^-(x_{n_2}^\omega, n_2) + C_1(\omega) e^{-(\lambda' - \epsilon)|n_2|} \\ &= C(n_2, \omega, \varphi) + C_1(\omega) e^{-(\lambda' - \epsilon)N/3} \leq C(n_2, \omega, \varphi) + e^{-(\lambda' - 2\epsilon)N/3} \end{aligned}$$

if N is large enough. Proposition 5.1 follows by taking $\lambda = \min\{\lambda_1, (\lambda' - 2\epsilon)/3\}$.

We now prove (6.9). Choose $\beta = \beta_0$ as in Lemma 6.5, and δ such that

$$\delta^{\frac{1}{8}} < \min\{\rho(\omega), \beta_0/2\}. \quad (6.10)$$

Using Proposition 2.1, there is $N_0(\omega)$ large enough such that for all $N_0 > N(\omega)$, all $\varphi \in C(\mathbb{T}^d)$, we have

$$\|K_{-N,n}^{\omega,b}\varphi - \psi_\omega^-(\cdot, n)\|_* < \delta, \quad -2N/3 \leq n \leq 0.$$

In particular, for any minimizer (y_n, η_n) , we have

$$(y_{j+n_2}, \eta_{j+n_2})_{j=-\bar{N}}^0 \text{ satisfies } \mathbf{AP}^-(\omega_2, \varphi_1, \delta, \bar{N}) \text{ condition.}$$

Apply Lemma 6.3, obtain

$$(y_{j+n_1}, \eta_{j+n_1})_{j=0}^{\bar{N}} \text{ satisfies } \mathbf{AP}^+(\omega_1, \check{\varphi}_2, \delta_1, \bar{N}) \text{ condition,}$$

with $\delta_1 = \max\{\delta^{q-\frac{1}{4}}, e^{-(\lambda'-5\epsilon)N/54}\}$.

Now we are going to apply Lemma 6.3 repeatedly, from \mathbf{AP}^- to \mathbf{AP}^+ and back until a desired estimate for δ is achieved. We shall assume that $N > N_2(\omega)$. On the first step we get an estimate $\mathbf{AP}^+(\omega_1, \varphi_2, \delta_1, \bar{N})$ for $(y_{j+n_2}, \eta_{j+n_2})_{j=-\bar{N}}^0$ where $\delta_1 = \max\{\delta^{q-\frac{1}{4}}, e^{-(\lambda'-5\epsilon)N/54}\}$. Since $q - \frac{1}{4} > 1$ this estimate is an improvement of δ unless $\delta < e^{-(\lambda'-5\epsilon)N/54}$. Notice that if this happens we have already proven our statement with $\lambda_1 = (\lambda' - 5\epsilon)/54$. It is easy to see that the level $\delta < e^{-(\lambda'-5\epsilon)N/54}$ will be reached in a finite number of steps depending on N . Notice N is large enough but fixed, this finishes the proof. \square

7. PROPERTIES OF THE FINITE TIME SOLUTIONS

We have proven all our statements except Proposition 6.1, which we prove in the next two sections.

7.1. The guiding orbit. For $N \in \mathbb{N}$, denote

$$\psi_\omega^N(x, n) = K_{-N,n}^{\omega,b}\varphi(x), \quad -N \leq n \leq 0.$$

We define

$$Q_\omega^N(x, n) = \psi_\omega^N(x, n) - \psi_\omega^+(x, n), \quad -N \leq n \leq 0, \quad (7.1)$$

which is a finite time analog of Q_ω^∞ . (Again, the subscript ω may be dropped).

The function Q^N is a Lyapunov function for minimizers, in the following sense:

Lemma 7.1. *Let $(y_n, \eta_n)_{n=-N}^0$ be a minimizer for $K_{-N,0}^{\omega,b}\varphi(y_0)$ (will use $\psi^N(x, 0)$ from now on). Then for all $-N \leq j < k \leq 0$,*

$$Q^N(y_j, j) \leq Q^N(y_k, k).$$

Proof. By definition,

$$\begin{aligned} Q^N(y_k, k) &= \psi^N(y_k, k) - \psi^+(y_k, k) = \psi^N(y_j, j) + A_{j,k}^{\omega,b}(y_j, y_k) - \psi^+(y_k, k) \\ &\geq \psi^N(y_j, j) - \left(\check{K}_{j,k}^{\omega,b}\psi^+(\cdot, k) \right)(y_j) = \psi^N(y_j, j) - \psi^+(y_j, j) = Q^N(y_j, j). \end{aligned}$$

\square

Let

$$z_{-N}^\omega \in \arg \min_z Q^N(z, -N),$$

and define $(z_n^\omega, \zeta_n^\omega)_{n=-N}^\infty$ to be a forward minimizer starting from z_{-N}^ω . The orbit $(z_n^\omega, \zeta_n^\omega)$ plays the role of the global minimizer (x_n^ω, v_n^ω) in the finite time set up, and is called the *guiding orbit*. The choice of z_{-N}^ω may not be unique, but our analysis will not depend on the choice of z_{-N}^ω .

Lemma 7.2. *The guiding orbit has the following properties.*

(1)

$$z_n^\omega \in \arg \min_z Q^N(z, n), \quad -N \leq n \leq 0.$$

(2)

$$\zeta_n^\omega = \nabla \psi^N(z_n^\omega, n) + b = \nabla \psi^+(z_n^\omega, n) + b, \quad -N \leq n \leq 0.$$

where both gradients exists.

(3)

$$Q^N(z_j^\omega, j) = Q^N(z_k^\omega, k), \quad -N \leq j < k \leq 0.$$

(4) $z_k^\omega, -N \leq k \leq 0$ is a backward minimizer for $K_{-N,0}^{\omega,b} \varphi$.

Proof. We prove (3) first. Since $(z_j^\omega)_{j \geq -N}$ is a forward minimizer, we have

$$\begin{aligned} Q^N(z_j^\omega, j) &= \psi^N(z_j^\omega, j) - \psi^+(z_j^\omega, j) = \psi^N(z_j^\omega, j) - \psi^+(z_k^\omega, k) + A_{j,k}^{\omega,b}(z_j^\omega, z_k^\omega) \\ &\geq \psi^N(z_k^\omega, k) - \psi^+(z_k^\omega, k) = Q^N(z_k^\omega, k). \end{aligned} \quad (7.2)$$

On other other hand, let $y_k \in \arg \min Q^N(\cdot, k)$, and let $(y_n)_{n=-N}^k$ be a minimizer for φ ending at y_k . Then by an argument similar to Lemma 7.1, for any $y \in \mathbb{T}^d$,

$$Q^N(y, k) \geq Q^N(y_k, k) \geq Q^N(y_{-N}, -N) \geq Q^N(z_{-N}^\omega, -N). \quad (7.3)$$

In particular, taking $y = z_k^\omega$, we have $Q^N(z_k^\omega, k) \geq Q^N(z_{-N}^\omega, -N)$. Using (7.2) for $j = -N$, we get $Q^N(z_{-N}^\omega, -N) = Q^N(z_k^\omega, k)$ which implies (3).

This also implies that (7.2) is in fact an equality, therefore $\psi^N(z_j^\omega, j) = \psi^N(z_k^\omega, k) - A_{j,k}^{\omega,b}(z_j^\omega, z_k^\omega)$ and (4) follows.

Using again (7.3), we have

$$\min_y Q^N(y, k) \geq Q^N(z_{-N}^\omega, -N) = Q^N(z_k^\omega, k)$$

which implies (1). Finally, since $\psi^N(\cdot, n)$ is a semi-concave function for $n \geq -N$ and ψ^+ is semi-convex, (2) follows from Lemma 4.2. \square

Combine (3) of Lemma 7.2 with Lemma 7.1, we get

$$Q^N(y_j, j) - Q^N(z_j^\omega, j) \leq Q^N(y_k, k) - Q^N(z_k^\omega, k). \quad (7.4)$$

7.2. Regular time and localization of the guiding orbit. We use the idea of regular time again. Let $0 < \beta < 1$ be such that

$$P(C(\omega) < \beta^{-1}, r(\omega) > \beta) > \frac{23}{24},$$

with $C(\omega), r(\omega)$ from Proposition 3.4. Let $M_0(\omega)$ be the random variable given by Proposition 5.2 with $\tilde{\rho} = \beta$. Let $\beta_1 > 0$ be such that $P(M_0(\omega) > \beta_1^{-1}) < 1/24$. n is called regular if

$$C(\theta^n \omega) < \beta^{-1}, \quad r(\theta^n \omega) > \beta, \quad M_0(\omega) < \beta_1^{-1},$$

using same proof as Lemma 6.5, we get

Lemma 7.3. *There exists $N_1(\omega) > 0$ such that for all $N > N_1(\omega)$, there exists a regular time in each time interval of size at least $N/12$ contained in $[-N, 0]$.*

Lemma 7.4. *There exists $N_4(\omega) > 0$ depending on β, ϵ such that for all $N > N_4(\omega)$, and $-5N/6 \leq k \leq 0$, we have*

$$\|z_k^\omega - x_k^\omega\| \leq \beta^{-1} e^{-\lambda'|k+5N/6|}.$$

By possibly enlarging $N_4(\omega)$, we have

$$0 \leq \psi^N(x_j^\omega) + A_{j,k}^{\omega,b}(x_j^\omega, x_k^\omega) - \psi^N(x_k^\omega) \leq e^{-(\lambda' - 3\epsilon)N/6}, \quad -2N/3 \leq j < k \leq 0.$$

Proof. The proof is again very similar to that of Lemma 6.5. Let n_* be a regular time in $[-5N/6, -11N/12]$. Then

$$\|z_{n_*}^\omega - x_{n_*}^\omega\| < \beta < r(\theta^{n_*} \omega).$$

Apply Proposition 3.3, we get

$$\|z_k^\omega - x_k^\omega\| \leq C(\theta^{n_*} \omega) e^{-\lambda'|k-n_*|} \leq \beta^{-1} e^{-\lambda'|k-n_*|}.$$

Since $n_* \geq -5N/6$, the first estimate follows. For the second estimate, since z_k^ω is a minimizer for $K_{-N,0}\varphi$, we have

$$0 = \psi^N(z_j^\omega) + A_{j,k}^{\omega,b}(z_j^\omega, z_k^\omega) - \psi^N(z_k^\omega), \quad -2N/3 \leq j < k \leq 0.$$

To avoid magnifying the coefficient of ϵ , let $K^{\epsilon/2} > K(\omega)$ be the result of applying Lemma 4.9 with parameter $\epsilon/2$, then

$$\begin{aligned} \psi^N(x_j^\omega) + A_{j,k}^{\omega,b}(x_j^\omega, x_k^\omega) - \psi^N(x_k^\omega) &\leq 2K^{\epsilon/2}(\theta^j \omega) \|z_j^\omega - x_j^\omega\| + 2K^{\epsilon/2}(\theta^k \omega) \|z_k^\omega - x_j^\omega\| \\ &\leq 4K^{\epsilon/2}(\omega) e^{\frac{\epsilon}{2} \cdot \frac{2N}{3}} e^{-\lambda'N/6} \leq 4K^{\epsilon/2}(\omega) e^{-(\lambda' - 2\epsilon)N/6} \leq e^{-(\lambda' - 3\epsilon)N/6}, \end{aligned}$$

where the last step is achieved by taking $N_4(\omega)$ large enough. \square

We now prove Lemma 6.4.

Proof of Lemma 6.4. We note that Lemma 7.4 proves half of the estimates in Lemma 6.4. The other half is proven using the same argument and reversing time. \square

For the rest of the paper, we will only deal with time $n \geq -N/3$. We record the improved estimates on this time interval.

Lemma 7.5. *There $N_4(\omega) > 0$ such that for $N > N_4(\omega)$, the following holds for $-N/3 \leq k \leq 0$: let f be any of the following functions: $\psi^N(\cdot, k)$, $\psi^\pm(\cdot, k)$, $A_{m,k}(y, \cdot)$ or $A_{k,n}(\cdot, y)$, \cdot . Then*

$$|f(x_k^\omega) - f(z_k^\omega)| \leq e^{-(\lambda-2\epsilon)N/3}.$$

Proof. We note that all choices of f are $K(\theta^k\omega)$ Lipschitz functions. Then

$$\begin{aligned} |f(x_k^\omega) - f(z_k^\omega)| &\leq K(\theta^k\omega) \|x_k^\omega - z_k^\omega\| \leq K^\epsilon(\theta^k\omega) \beta^{-1} e^{-\lambda'N/3} \\ &\leq K^\epsilon(\omega) e^{\epsilon|k|} \beta^{-1} e^{-\lambda'N/3} \leq \beta^{-1} K^\epsilon(\omega) e^{-(\lambda'-\epsilon\epsilon)N/3} \leq e^{-(\lambda'-2\epsilon)N/3} \end{aligned}$$

for N large enough. \square

7.3. Stability of the finite time minimizers. We show that if an orbit $(y_n, \eta_n)_{n=-N}^0$ satisfies $\mathbf{AP}^-(\omega, \varphi, \delta, N)$ condition, then it is stable in the backward time. First, we obtain an analog of (4.1).

Lemma 7.6. *Assume the orbit (y_n, η_n) satisfies $\mathbf{AP}^-(\omega, \varphi, \delta, N)$, Then for each $-N/3 \leq k \leq 0$ such that $\|y_k - x_k^\omega\| < r(\theta^k\omega)$, we have*

$$\begin{aligned} Q^N(y_k, k) - Q^N(x_k^\omega, k) &\geq a(\theta^k\omega) \|y_k - x_k^\omega\|^2 - \delta, \\ Q^N(y_k, k) - Q^N(x_k^\omega, k) &\leq K(\theta^k\omega) \|y_k - x_k^\omega\|^2 + \delta. \end{aligned}$$

Proof. The definition of \mathbf{AP}^- implies that for all $\|y_k - x_k^\omega\| < r(\theta^k\omega)$,

$$\| (Q^N(y_k, k) - Q^N(x_k^\omega, k)) - (Q^\infty(y_k, k) - Q^\infty(x_k^\omega, k)) \| < \delta$$

the lemma follows directly from (4.1). \square

We combine this with (7.4) to obtain a backward stability for (y_n, η_n) .

Lemma 7.7. *Assume that (y_n, η_n) satisfies $\mathbf{AP}^-(\omega, \varphi, \delta, N)$ with*

$$\delta \geq e^{-(\lambda'-3\epsilon)N/3}.$$

There exists $C_1^\epsilon(\omega) > 0$ with $e^{-\epsilon} \leq C_1^\epsilon(\omega)/C_1^\epsilon(\theta\omega) \leq e^\epsilon$ such that, if for $N > N_4(\omega)$, minimizer, if $-N/3 \leq j < k \leq 0$, satisfies $\|y_j - x_j^\omega\| < r^\epsilon(\theta^j\omega)$, $\|y_k - x_k^\omega\| < r^\epsilon(\theta^k\omega)$, we have

$$\|y_j - x_j^\omega\| \leq C_1^\epsilon(\theta^k\omega) e^{\epsilon|j-k|} \left(\|y_k - x_k^\omega\| + 3\sqrt{\delta} \right).$$

Proof. Apply Lemma 7.5, we have

$$\begin{aligned} Q^N(y_j, j) - Q^N(x_j^\omega, j) &\leq Q^N(y_j, j) - Q^N(z_j^\omega, j) + 2e^{-(\lambda'-3\epsilon)N/3} \\ &\leq Q^N(y_k, k) - Q^N(z_k^\omega, k) + 2e^{-(\lambda'-3\epsilon)N/3} \\ &\leq Q^N(y_k, k) - Q^N(x_k^\omega, k) + 4e^{-(\lambda'-3\epsilon)N/3} \\ &\leq Q^N(y_k, k) - Q^N(x_k^\omega, k) + 4\delta. \end{aligned}$$

Combine with Lemma 7.6, we get

$$a^\epsilon(\theta^j\omega) \|y_j - x_j^\omega\|^2 \leq K^\epsilon(\theta^k\omega) \|y_k - x_k^\omega\|^2 + 6\delta.$$

Using $a^\epsilon(\theta^j\omega) \geq e^{-\epsilon|j-k|}a^\epsilon(\theta^k\omega)$, we obtain

$$\|y_j - x_j^\omega\|^2 \leq K^\epsilon/a^\epsilon(\theta^k\omega)e^{\epsilon|j-k|}(\|y_k - x_k^\omega\|^2 + 6\delta),$$

therefore

$$\|y_j - x_j^\omega\| \leq e^{\epsilon|j-k|}\sqrt{K^\epsilon/a^\epsilon(\theta^k\omega)}(\|y_k - x_k^\omega\| + 3\sqrt{\delta}).$$

The lemma follows by taking $C_1^\epsilon = \sqrt{K^\epsilon/a^\epsilon}$. \square

8. ESTIMATES FROM NON-UNIFORM HYPERBOLICITY

8.1. Hyperbolic properties of the global minimizer. Denote

$$X_n^\omega = (x_n^\omega, \zeta_n^\omega) \in \mathbb{T}^d \times \mathbb{R}^d,$$

this orbit is non-uniformly hyperbolic.

Proposition 8.1. *For any $\epsilon > 0$, the following hold.*

- (1) *(Stable and unstable bundles) For each $n \in \mathbb{Z}$, there exists the splitting*

$$\mathbb{R}^{2d} = E^s(X_n^\omega) \oplus E^u(X_n^\omega),$$

where $\dim E^s = \dim E^u = d$. We denote by Π_n^s, Π_n^u the projection to E^s, E^u under this splitting.

- (2) *(Lyapunov norm) There exist norms $\|\cdot\|_n^s, \|\cdot\|_n^u$ on \mathbb{R}^d , and the Lyapunov norm on \mathbb{R}^{2d} is defined by*

$$(\|v\|_n')^2 = (\|\Pi_n^s v\|_n^s)^2 + (\|\Pi_n^u v\|_n^u)^2.$$

There exists a function $M^\epsilon(\omega) > 0$ satisfying $e^{-\epsilon} \leq M^\epsilon(\omega)/M^\epsilon(\theta\omega) \leq e^\epsilon$ such that

$$\|v\| \leq \|v\|_n' \leq M^\epsilon(\theta^n\omega)\|v\|,$$

where $\|\cdot\|$ is the Euclidean metric. We will omit the subscript from the $\|\cdot\|_n'$ and $\Pi_n^{s/u}$ when the index is clear from context.

- (3) *(Cones) We define the unstable cones*

$$C_n^u = \{v \in \mathbb{R}^{2d} : \|\Pi_n^s v\|_n^s \leq \|\Pi_n^u v\|_n^u\},$$

and the stable cones

$$C_n^s = \{v \in \mathbb{R}^{2d} : \|\Pi_n^u v\|_n^u \leq \|\Pi_n^s v\|_n^s\}.$$

- (4) *(Hyperbolicity) There exists $\sigma^\epsilon(\omega) > 0$ with $e^{-\epsilon} \leq \sigma^\epsilon(\omega)/\sigma^\epsilon(\theta\omega) \leq e^\epsilon$, such that the following hold. Let Y_n be an orbit of Φ_n^ω .*

(a) *If*

$$\|Y_n - X_n\|' < \sigma^\epsilon(\theta^n\omega), \quad \|Y_{n-1} - X_{n-1}\|' < \sigma^\epsilon(\theta^{n-1}\omega),$$

then

$$\|\Pi^s Y_{n-1} - \Pi^s X_{n-1}\|' \geq e^{\lambda'} \|\Pi^s Y_n - \Pi^s X_n\|',$$

where $\lambda' = \lambda - \epsilon$. Moreover, if $Y_n - X_n \in C_n^s$, then $Y_{n-1} - X_{n-1} \in C_{n-1}^s$. In other words, the stable cones are backward invariant and backward expanding.

(b) *If*

$$\|Y_n - X_n\|' < \sigma^\epsilon(\theta^n \omega), \quad \|Y_{n-1} - X_{n-1}\|' < \sigma^\epsilon(\theta^{n-1} \omega),$$

then

$$\|\Pi^u Y_{n-1} - \Pi^u X_{n-1}\|' \leq e^{-\lambda'} \|\Pi^u Y_n - \Pi^u X_n\|'.$$

Moreover, if $Y_n - X_n \in C_n^u$, then $Y_{n-1} - X_{n-1} \in C_n^u$.

8.2. Stability of minimizer in the phase space. We improve Lemma 7.7 to its counter part in the phase space, using the Lyapunov norm.

Lemma 8.2. *Under the same assumption of Lemma 7.7, there exists $C_2^\epsilon(\omega) > 0$ and $e^\epsilon \leq C_2^\epsilon(\omega)/C_2^\epsilon(\theta\omega) \leq e^{-\epsilon}$ such that, if $-N/3 < j < k \leq 0$ satisfies*

$$\|y_j - x_j^\omega\| < r(\theta^j \omega), \quad \|y_k - x_k^\omega\| < r(\theta^k \omega),$$

then

$$\|Y_j - X_j^\omega\|' \leq C_2^\epsilon(\theta^k \omega) e^{\epsilon|j-k|} \left(\|y_k - x_k^\omega\| + \sqrt{\delta} \right).$$

Proof. Apply Lemma 7.7, we have

$$\|y_j - x_j^\omega\| \leq C_1^\epsilon(\theta^k \omega) e^{\epsilon|j-k|} \left(\|y_k - z_k^\omega\| + 3\sqrt{\delta} \right).$$

Since $j < -1$, we use Lemma 4.2 to get

$$\|\eta_j - v_j^\omega\| \leq K^\epsilon(\theta^j \omega) \|y_j - x_j^\omega\|,$$

and hence

$$\|Y_j - X_j^\omega\| \leq 2K^\epsilon(\theta^j \omega) \|y_j - x_j^\omega\|.$$

We now have

$$\begin{aligned} \|Y_j - X_j^\omega\|' &\leq M^\epsilon(\theta^j \omega) \|Y_j - X_j\| \leq 2(M^\epsilon K^\epsilon)(\theta^j \omega) \|y_j - x_j^\omega\| \\ &\leq 2C_1^\epsilon(\theta^k \omega) (M^\epsilon K^\epsilon)(\theta^k \omega) e^{2\epsilon|j-k|} \left(\|y_k - x_k^\omega\| + 3\sqrt{\delta} \right) \\ &\leq 6C_1^\epsilon(\theta^k \omega) (M^\epsilon K^\epsilon)(\theta^k \omega) e^{\epsilon|j-k|} \left(\|y_k - x_k^\omega\| + \sqrt{\delta} \right). \end{aligned}$$

We now replace ϵ with $\epsilon/3$ in the above estimate, and define $C_2^\epsilon = 6C_1^{\epsilon/3} M^{\epsilon/3} K^{\epsilon/3}$, which satisfies $e^\epsilon \leq C_2^\epsilon(\omega)/C_2^\epsilon(\theta\omega) \leq e^{-\epsilon}$. The lemma follows. \square

8.3. Exponential localization using hyperbolicity. We now show hyperbolicity, together with Lemma 8.2 lead to a stronger localization. In Lemma 8.3 we show a dichotomy: either $y_n - x_n^\omega$ contracts for each backward iterate, or y_n is δ^q close to x_n to begin with.

Define

$$r_1(\omega) = \min\{r^{\epsilon/4}(\omega), \sigma^{\epsilon/4}(\omega)\},$$

where r^ϵ is defined in Lemma 4.9, and σ^ϵ defined in property (4) of Proposition 8.1. Now define

$$\rho_0(\omega) = \left(\frac{r_1(\omega)}{3C_2^{\epsilon/4}(\omega)} \right)^2,$$

then $e^\epsilon \leq \rho_0(\omega)/\rho_0(\theta\omega), r_1(\omega)/r_1(\theta\omega) \leq e^\epsilon$.

Lemma 8.3. *Let $(y_n, \eta_n)_{n=-N}^0$ be the minimizer as before. Suppose for a given $-N/6 \leq k \leq -1$, we have*

$$\|Y_k - X_k\| < \rho_0(\theta^k\omega), \quad \delta^{\frac{1}{4}} < \rho_0(\theta^k\omega). \quad (8.1)$$

Then one of the following alternatives must hold for Y_k :

(1) $Y_k - X_k^\omega \in C_k^s$ and

$$\|Y_k - X_k^\omega\|' \leq \max\{\delta^q, e^{-(\lambda' - 2\delta)N/6}\}, \quad q = \lambda'/(8\epsilon). \quad (8.2)$$

(2) $Y_k - X_k^\omega \in C_k^u$ and

$$\|\Pi^u Y_{k-1} - \Pi^u X_{k-1}\|' \leq e^{-\lambda'} \|\Pi^u Y_k - \Pi^u X_k\|. \quad (8.3)$$

Proof. Let us denote $d_k = \|Y_k - X_k^\omega\|'$, $\bar{\delta} = \sqrt{\delta}$ and $C_3^\epsilon = C_2^{\epsilon/4}$. Define $-N/6 \leq i_0 < 0$ by the relation

$$\begin{aligned} e^{2\epsilon|i_0|} &= \min \left\{ \frac{r_1(\theta^k\omega)}{3d_k C_3^\epsilon(\theta^k\omega)}, \frac{r_1(\theta^k\omega)}{3\bar{\delta} C_3^\epsilon(\theta^k\omega)}, e^{\epsilon N/3} \right\} \\ &= \min \left\{ \frac{\sqrt{\rho_0}(\theta^k\omega)}{d_k}, \frac{\sqrt{\rho_0}(\theta^k\omega)}{\bar{\delta}}, e^{\epsilon N/3} \right\} \end{aligned} \quad (8.4)$$

In particular, we have

$$C_3^\epsilon(\theta^k\omega) e^{2\epsilon|i_0|} \leq \min \left\{ \frac{r_1(\theta^k\omega)}{3d_k}, \frac{r_1(\theta^k\omega)}{3\bar{\delta}} \right\} \quad (8.5)$$

and (using $C_3^\epsilon = \frac{r_1}{3\sqrt{\rho_0}}$ and (8.1))

$$C_3^\epsilon(\theta^k\omega) e^{-2\epsilon|i_0|} \leq \max \left\{ \frac{d_k r_1(\theta^k\omega)}{3\rho_0(\theta^k\omega)}, \frac{\bar{\delta} r_1(\theta^k\omega)}{3\rho_0(\theta^k\omega)} \right\} < \frac{r_1(\theta^k\omega)}{3} < 1. \quad (8.6)$$

Since $-N/3 \leq i_0 + k < 0$, Lemma 8.2 applies. Using (8.5), we have for each $i_0 + k \leq j < k$,

$$\begin{aligned} d_j &\leq C_3^\epsilon(\theta^k\omega) e^{\epsilon|j-k|} (d_k + \bar{\delta}) = e^{-\epsilon|i_0|} C_3^\epsilon(\theta^k\omega) e^{2\epsilon|i_0|} (d_k + \bar{\delta}) \\ &\leq e^{-\epsilon|i_0|} (d_k + \bar{\delta}) \min \left\{ \frac{r_1(\theta^k\omega)}{3d_k}, \frac{r_1(\theta^k\omega)}{3\bar{\delta}} \right\} \leq e^{-\epsilon|i_0|} \cdot \frac{2}{3} r_1(\theta^k\omega) < r_1(\theta^j\omega). \end{aligned} \quad (8.7)$$

As a result, Proposition 8.1 (4) applies for $i_0 + k \leq j < k$.

If $Y_k - X_k^\omega \in C_k^u$ (second alternative), (8.3) holds by Proposition 8.1 (4)(b). If $Y_k - X_k^\omega \in C_k^s$ (first alternative), then $Y_j - X_j^\omega \in C_j^s$ for all $i_0 + k < j < k$, due to backward invariance of stable cones. As a result, we get from Proposition 8.1 (4)(a) that

$$d_j \geq e^{\lambda'|j-k|} d_k.$$

Pick $j = i_0 + k$, combine with the first line of (8.7), and using (8.6), we get

$$\begin{aligned} d_k &\leq C_3^\epsilon(\theta^k\omega) e^{-(\lambda' - \epsilon)|i_0|} (d_k + \bar{\delta}) = C_3^\epsilon(\theta^k\omega) e^{-2\epsilon|i_0|} e^{-(\lambda' - 3\epsilon)|j-k|} (d_k + \bar{\delta}) \\ &\leq e^{-(\lambda' - 3\epsilon)|i_0|} (d_k + \bar{\delta}). \end{aligned}$$

We can choose ϵ small enough (and as a result $|i_0|$ large enough) such that $e^{-(\lambda'-3\epsilon)|i_0|} < \frac{1}{2}$, then

$$\bar{\delta}e^{-3(\lambda'-3\epsilon)|i_0|} \geq d_k(1 - e^{-3(\lambda'-3\epsilon)|i_0|}) \geq \frac{1}{2}d_k.$$

Note that in this case, $d_k \leq \bar{\delta}$. As a result, using (8.4), and $\bar{\delta}^{\frac{1}{2}} < \rho(\theta^k \omega)$ from (8.1), we get

$$e^{-2\epsilon|i_0|} = \max \left\{ \frac{\bar{\delta}}{\sqrt{\rho_0(\theta^k \omega)}}, \quad e^{-\epsilon N/6} \right\} = \max \{ \bar{\delta}^{\frac{1}{2}}, \quad e^{-\epsilon N/3} \} = \max \{ \delta^{\frac{1}{4}}, \quad e^{-\epsilon N/3} \}.$$

We combine with the previous formula to get

$$d_k \leq 2\bar{\delta} \left(e^{-2\epsilon|i_0|} \right)^{(\lambda'-3\epsilon)/(2\epsilon)} \leq \left(e^{-2\epsilon|i_0|} \right)^{(\lambda'-3\epsilon)/(2\epsilon)} \leq \max \{ \delta^q, \quad e^{-(\lambda'-3\epsilon)N/6} \}$$

with $q = (\lambda' - 3\epsilon)/(8\epsilon)$. \square

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. Define

$$\rho(\omega) = \rho_0(\omega)/(4C_3^\epsilon(\omega)) < \rho_0(\omega),$$

then $e^{-2\epsilon} \leq \rho(\theta\omega)/\rho(\omega) \leq e^{2\epsilon}$. Recall the assumption

$$\delta^{\frac{1}{8}} < \rho(\omega).$$

Denote $k_0 = \max \{ \frac{1}{8\epsilon} \log \delta, -\frac{N}{6} \}$, we have

$$e^{-\epsilon|k_0|} = e^{\epsilon k_0} \geq e^{\frac{1}{8} \log \delta} = \delta^{\frac{1}{8}}$$

and

$$\rho_0(\theta^k \omega) \geq e^{-\epsilon|k_0|} \rho_0(\omega) \geq \delta^{\frac{1}{4}}, \quad k_0 \geq k \geq 0.$$

If $\|y_0 - x_0^\omega\| < \rho(\omega)$, by Lemma 8.2, we have

$$\|Y_{-1} - X_{-1}^\omega\|' \leq C_3^\epsilon(\omega)e^\epsilon \left(\|y_0 - x_0^\omega\| + \sqrt{\delta} \right) \leq C_3^\epsilon(\omega)e^\epsilon 2\rho(\omega) = e^\epsilon \rho_0(\omega)/2.$$

Assume $e^\epsilon < 2$, then Lemma 8.3 applies for $k = -1$. If alternative (8.2) hold, we obtain $\|Y_k - X_k^\omega\|' \leq \max \{ \delta^q, e^{-(\lambda'-3\epsilon)N/6} \}$ and the proposition follows. Otherwise, the alternative (8.3) applies, and

$$\begin{aligned} \|Y_{-2} - X_{-2}^\omega\|' &\leq \sqrt{2}\|\Pi^s(Y_{-2} - X_{-2}^\omega)\|^s \leq \sqrt{2}e^{-\lambda'}\|Y_{-1} - X_{-1}\|' \\ &< e^{-\lambda'}e^\epsilon/\sqrt{2}\rho_0(\epsilon) < e^{-\lambda'}\rho_0(\omega) \leq \rho_0(\theta^{-1}\omega). \end{aligned}$$

if $e^\epsilon < \sqrt{2}$. We can apply Lemma 8.3 again. Suppose (8.2) does not hold for $k+1 \leq j \leq -1$. Then

$$\|Y_k - X_k^\omega\|' \leq \sqrt{2}e^{-|k+1|\lambda'}\|Y_{-1} - X_{-1}\|' < e^{-|k+1|\lambda'}\rho_0(\omega) \leq \rho_0(\theta^k \omega).$$

Therefore this argument can be applied inductively until we reach $k = k_0$, then

$$\begin{aligned} \|y_{k_0} - x_{k_0}^\omega\| &\leq \|Y_{k_0} - X_{k_0}^\omega\|' \leq \sqrt{2}e^{-\lambda'|k_0|} \leq \sqrt{2}\rho_0(\omega) \max \left\{ \delta^{\frac{\lambda'}{8\epsilon}}, e^{-\lambda'N/6} \right\} \\ &< \max \{ \delta^q, e^{-(\lambda'-3\epsilon)N/6} \}. \end{aligned}$$



REFERENCES

- [1] L. Barreira and Y. Pesin. *Nonuniform Hyperbolicity, Encyclopedia of Mathematics and Its Applications 115*. Cambridge University Press, 2007.
- [2] G. Contreras, A. Figalli, and L. Rifford. Generic hyperbolicity of Aubry sets on surfaces. *Inventiones Mathematicae*, 200(1):201–261, 2015.
- [3] Gonzalo Contreras. Generic mañé sets. *Preprint, arxiv:1410.7141*.
- [4] Weinan E, K. Khanin, A. Mazel, and Ya. Sinai. Invariant measures for Burgers equation with stochastic forcing. *Ann. of Math. (2)*, 151(3):877–960, 2000.
- [5] Albert Fathi. *Weak KAM theorem in Lagrangian dynamics, 10th preliminary version*. book preprint, 2008.
- [6] R. Iturriaga and K. Khanin. Burgers turbulence and random Lagrangian systems. *Comm. Math. Phys.*, 232(3):377–428, 2003.
- [7] Renato Iturriaga and Héctor Sánchez-Morgado. Hyperbolicity and exponential convergence of the Lax-Oleinik semigroup. *Journal of Differential Equations*, 246(5):1744–1753, 2009.
- [8] Konstantin Khanin and Ke Zhang. Hyperbolicity of minimizers and regularity of viscosity solutions for a random hamilton-jacobi equation. *Preprint, arXiv:1201.3381*.
- [9] Ricardo Mañé. Lagrangian flows: the dynamics of globally minimizing orbits. *Boletim da Sociedade Brasileira de Matemática. Nova Série*, 28(2):141–153, 1997.
- [10] John N. Mather. Variational construction of connecting orbits. *Ann. Inst. Fourier (Grenoble)*, 43(5):1349–1386, 1993.
- [11] Ya G Sinai. Two results concerning asymptotic behavior of solutions of the burgers equation with force. *Journal of statistical physics*, 64(1-2):1–12, 1991.

CIMAT, GUANAJUATO, MEXICO
E-mail address: `renato@cimat.mx`

UNIVERSITY OF TORONTO
E-mail address: `khanin@math.toronto.edu`

UNIVERSITY OF TORONTO
E-mail address: `kzhang@math.toronto.edu`